ORIGINAL ARTICLE

The theory of canonical perturbations applied to attitude dynamics and to the Earth rotation. Osculating and nonosculating Andoyer variables

Michael Efroimsky · Alberto Escapa

Abstract   In the method of variation of parameters we express the Cartesian coordinates or the Euler angles as functions of the time and six constants. If, under disturbance, we endow the "constants" with time dependence, the perturbed orbital or angular velocity will consist of a partial time derivative and a convective term that includes time derivatives of the "constants". The Lagrange constraint, often imposed for convenience, nullifies the convective term and thereby guarantees that the functional dependence of the velocity on the time and "constants" stays unaltered under disturbance. "Constants" satisfying this constraint are called osculating elements. Otherwise, they are simply termed orbital or rotational elements. When the equations for the elements are required to be canonical, it is normally the Delaunay variables that are chosen to be the orbital elements, and it is the Andoyer variables that are typically chosen to play the role of rotational elements. (Since some of the Andoyer elements are time-dependent even in the unperturbed setting, the role of "constants" is actually played by their initial values.) The Delaunay and Andoyer sets of variables share a subtle peculiarity: under certain circumstances the standard equations render the elements nonosculating. In the theory of orbits, the planetary equations yield nonosculating elements when perturbations depend on velocities. To keep the elements osculating, the equations must be amended with extra terms that are *not* parts of the disturbing function [Efroimsky, M., Goldreich, P.: J. Math. Phys. **44**, 5958–5977 (2003); Astron. Astrophys. **415**, 1187–1199 (2004); Efroimsky, M.: Celest. Mech. Dyn. Astron. **91**, 75–108 (2005); Ann. New York Acad. Sci. **1065**, 346–374 (2006)]. It complicates both the Lagrange- and Delaunay-type planetary equations and makes the Delaunay equations noncanonical. In attitude dynamics, whenever a perturbation depends upon the angular velocity (like a switch to a noninertial frame), a mere amendment of the Hamiltonian makes the equations yield nonosculating Andoyer elements. To make them osculating, extra terms should be added to the equations (but then the equations will no

M. Efroimsky (✉)
US Naval Observatory, Washington, DC 20392, USA
e-mail: me@usno.navy.mil

A. Escapa
Departamento de Matemática Aplicada, Universidad de Alicante,
Alicante E-03080, Spain

Report Documentation Page

1. REPORT DATE **14 APR 2006**	2. REPORT TYPE	3. DATES COVERED **16-03-2007 to 02-04-2007**

4. TITLE AND SUBTITLE **The Theory of Canonical Perturbations Applied to Attitude Dynamics and to the Earth Rotation. Osculating and Nonosculating Andoyer Variables**	5a. CONTRACT NUMBER
	5b. GRANT NUMBER
	5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S)	5d. PROJECT NUMBER
	5e. TASK NUMBER
	5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) **US Naval Observatory ,Washington,DC,20392**	8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	10. SPONSOR/MONITOR'S ACRONYM(S)
	11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution unlimited

13. SUPPLEMENTARY NOTES

14. ABSTRACT

15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT **unclassified**	b. ABSTRACT **unclassified**	c. THIS PAGE **unclassified**	**Same as Report (SAR)**	**33**	

longer be canonical). Calculations in nonosculating variables are mathematically valid, but their physical interpretation is not easy. Nonosculating orbital elements parameterise instantaneous conics *not* tangent to the orbit. (A nonosculating *i* may differ much from the real inclination of the orbit, given by the osculating *i*.) Nonosculating Andoyer elements correctly describe perturbed attitude, but their interconnection with the angular velocity is a nontrivial issue. The Kinoshita–Souchay theory tacitly employs nonosculating Andoyer elements. For this reason, even though the elements are introduced in a precessing frame, they nevertheless return the inertial velocity, not the velocity relative to the precessing frame. To amend the Kinoshita–Souchay theory, we derive the precessing-frame-related directional angles of the angular velocity relative to the precessing frame. The loss of osculation should not necessarily be considered a flaw of the Kinoshita–Souchay theory, because in some situations it is the inertial, not the relative, angular velocity that is measurable [Schreiber, K. U. et al.: J. Geophys. Res. **109**, B06405 (2004); Petrov, L.: Astron. Astrophys. **467**, 359–369 (2007)]. Under these circumstances, the Kinoshita–Souchay formulae for the angular velocity should be employed (as long as they are rightly identified as the formulae for the *inertial* angular velocity).

1 The Hamiltonian approach to rotational dynamics

1.1 Historical preliminaries

The perturbed rotation of a rigid body has long been among the key topics of both spacecraft engineering (Giacaglia and Jefferys 1971; Zanardi and Vilhena de Moraes 1999) and planetary astronomy (Kinoshita 1977; Laskar and Robutel 1993; Touma and Wisdom 1994; Mysen 2004, 2006). While free spin (the Euler–Poinsot problem) permits an analytic solution in terms of the elliptic Jacobi functions, perturbed motion typically requires numerical treatment, though sometimes it can be dealt with by analytical means (like, for example, in Kinoshita 1977). Perturbation may come from a physical torque, or from an inertial torque caused by the frame noninertiality, or from nonrigidity (Getino and Ferrándiz 1990; Escapa et al. 2001, 2002). The free-spin Hamiltonian, expressed through the Euler angles and their conjugate momenta, is independent of one of the angles, which reveals an internal symmetry of the problem. In fact, this problem possesses an even richer symmetry (Deprit and Elipe 1993), whose existence indicates that the unperturbed Euler–Poinsot dynamics can be reduced to one degree of freedom. The possibility of such reduction is not readily apparent and can be seen only under certain choices of variables. These variables, in analogy with the orbital mechanics, are called rotational elements. It is convenient to treat the forced-rotation case as a perturbation expressed through those elements.

The Andoyer variables are often chosen as rotational elements (Andoyer 1923; Giacaglia and Jefferys 1971; Kinoshita 1972), though other sets of canonical elements have appeared in the literature (Richelot 1850; Serret 1866; Peale 1973, 1976; Deprit and Elipe 1993;

Fukushima and Ishizaki 1994).[1] After a transition to rotational elements is performed within the undisturbed Euler–Poinsot setting, the next step is to extend this method to a forced-rotation case. To this end, one will have to express the torques via the elements. On completion of the integration, one will have to return back from the elements to the original, measurable, quantities—i.e., to the Euler angles and their time derivatives.

1.2 The Kinoshita–Souchay theory of rigid-Earth rotation

The Hamiltonian approach to spin dynamics has found its most important application in the theory of Earth rotation. A cornerstone work on this topic was carried out by Kinoshita (1977) who switched from the Euler angles defining the Earth orientation to the Andoyer variables, and treated their dynamics by means of the Hori (1966) and Deprit (1969) technique.[2] Then he translated the results of this development back into the language of Euler's angles and provided the precessional and nutation spectrum. Later his approach was extended to a much higher precision by Kinoshita and Souchay (1990) and Souchay et al. (1999).

1.3 Subtle points

When one is interested only in the orientation of the rotator, it is sufficient to have expressions for the Euler angles as functions of the elements. However, when one needs to know also the instantaneous angular velocity, one needs expressions for the Euler angles' time derivatives. This poses the following question: if we write down the expressions for the Euler angles' derivatives via the canonical elements in the free-spin case, will these expressions stay valid under perturbation? In the parlance of orbital mechanics, this question may be formulated like this: are the canonical elements always osculating? As we shall demonstrate below, under angular-velocity-dependent disturbances the condition of osculation is incompatible with that of canonicity, and therefore expression of the angular velocity via the canonical elements will, under such types of perturbations, become nontrivial.

In 2004 the question acquired a special relevance to the Earth-rotation theory. While the thitherto available observations referred to the orientation of the Earth figure (Kinoshita et al. 1978), a technique based on ring laser gyroscope provided a direct measurement of the instantaneous angular velocity of the Earth relative to an inertial frame (Schreiber et al. 2004; Petrov 2007).

Normally, rotational elements are chosen to have evident physical interpretation. For example, the Andoyer variable G coincides with the absolute value of the body's spin angular momentum, while two other variables, H and L, are chosen to coincide, correspondingly, with the Z-component of the angular momentum in the inertial frame, and with its z-component in the body frame. The other Andoyer elements, g, l, h, too, bear some evident meaning. Hence another important question: will the canonical rotational elements preserve their simple physical meaning also under disturbance?

---

[1] Some authors use the term "Serret–Andoyer elements." This is not correct, because the set of elements introduced by Richelot (1850) and Serret (1866) differs from the one employed by Andoyer (1923).

[2] For an introduction into the Hori–Deprit method see Boccaletti and Pucacco (2002) and Kholshevnikov (1973, 1985). Kinoshita (1977) referred only to the work by Hori (1966).

2 The canonical perturbation theory in orbital and attitude dynamics

2.1 Kepler and Euler

In orbital dynamics, a Keplerian conic, emerging as an undisturbed two-body orbit, is regarded to be a "simple motion," so that all the other available motions are conveniently considered as distortions of such conics, distortions implemented through endowing the orbital constants C_j with their own time dependence. Points of the orbit can be contributed by the "simple curves" either in *a* nonosculating fashion, as in Fig. 1, or in *the* osculating manner, as in Fig. 2.

The disturbances, causing the evolution of the motion from one instantaneous conic to another, are the primary's nonsphericity, the gravitational pull of other bodies, the atmospheric and radiation-caused drag, the relativistic corrections, and the noninertiality of the reference system.

On Fig. 1 the orbit consists of points, each of which is donated by a representative of a certain family of "simple" curves (confocal ellipses). These instantaneous ellipses are *not* supposed to be tangent or even coplanar to the orbit. As a result, the physical velocity $\dot{\vec{r}}$ (tangent to the orbit) differs from the Keplerian velocity $\vec{g}$ (tangent to the ellipse). To parameterise the depicted sequence of nonosculating ellipses, and to single it out of the other sequences, it is suitable to employ the difference between $\dot{\vec{r}}$ and $\vec{g}$, expressed as a function of the time and the orbital elements: $\vec{\Phi}(t, C_1, \ldots, C_6) = \dot{\vec{r}}(t, C_1, \ldots, C_6) - \vec{g}(t, C_1, \ldots, C_6)$. Evidently,

$$\dot{\vec{r}} = \frac{\partial \vec{r}}{\partial t} + \sum_{j=1}^{6} \frac{\partial C_j}{\partial t} \dot{C}_j = \vec{g} + \vec{\Phi},$$
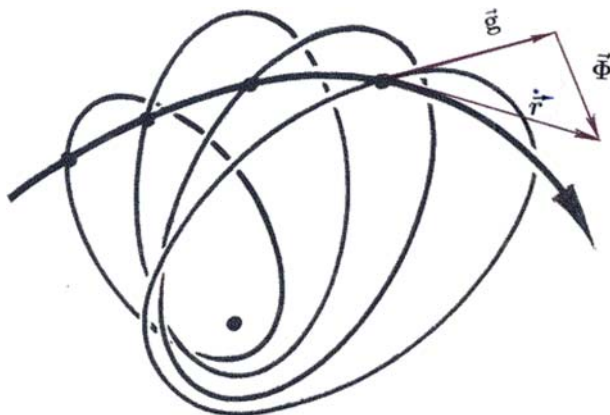
Fig. 1 The perturbed orbit is a set of points belonging to a sequence of confocal instantaneous ellipses that are *not* supposed to be tangent or even coplanar to the orbit. As a result, the physical velocity $\dot{\vec{r}}$ (tangent to the orbit) differs from the Keplerian velocity $\vec{g}$ (tangent to the ellipse). To parameterise the depicted sequence of nonosculating ellipses, and to single it out of the other sequences, it is suitable to employ the difference between $\dot{\vec{r}}$ and $\vec{g}$, expressed as a function of time and six (nonosculating) orbital elements: $\vec{\Phi}(t, C_1, \ldots, C_6) = \dot{\vec{r}}(t, C_1, \ldots, C_6) - \vec{g}(t, C_1, \ldots, C_6)$. In the literature, $\vec{\Phi}(t, C_1, \ldots, C_6)$ is called gauge function or gauge velocity or, simply, gauge
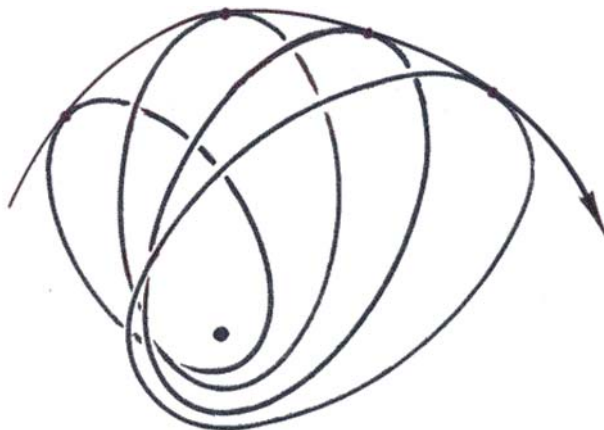
Fig. 2 The orbit is represented by a sequence of confocal instantaneous ellipses that are tangent to the orbit, i.e., osculating. Now, the physical velocity $\dot{\vec{r}}$ (tangent to the orbit) coincides with the Keplerian velocity $\vec{g}$ (tangent to the ellipse), so that their difference vanishes everywhere: $\vec{\Phi}(t, C_1, \ldots, C_6) = 0$. This is the so-called Lagrange constraint or Lagrange gauge. Orbital elements obeying it are called osculating

i.e., the unperturbed Keplerian velocity is $\vec{g} \equiv \partial \vec{r}/\partial t$, while the said difference $\vec{\Phi}$ is the convective term that emerges when the instantaneous ellipses are being gradually altered by the perturbation (and when the orbital elements become time-dependent): $\vec{\Phi} = \sum \left(\partial \vec{r}/\partial C_j\right) \dot{C}_j$. When one fixes a particular functional dependence of $\vec{\Phi}$ upon time and the elements, this function, $\vec{\Phi}(t, C_1, \ldots, C_6)$, is called gauge function or gauge velocity or, simply, gauge.

On Fig. 2, the perturbed orbit is represented with a sequence of confocal instantaneous ellipses that are tangent to the orbit, i.e., osculating. Under this choice, the physical velocity $\dot{\vec{r}}$ (tangent to the orbit) will coincide with the Keplerian velocity $\vec{g}$ (tangent to the ellipse), so that their difference $\vec{\Phi}(t C_1, \ldots, C_6)$ will vanish everywhere:

$$\vec{\Phi}(t, C_1, \ldots, C_6) \equiv \dot{\vec{r}}(t, C_1, \ldots, C_6) - \vec{g}(t, C_1, \ldots, C_6) = \sum_{j=1}^{6} \frac{\partial C_j}{\partial t} \dot{C}_j = 0.$$

This, so-called Lagrange constraint or Lagrange gauge, is the necessary and sufficient condition of osculation of the orbital elements C_j (Brouwer and Clemence 1961). Historically, the first attempt of using nonosculating elements dates back to Poincare (1897), though he never explored them from the viewpoint of a non-Lagrange constraint choice. (See also Abdullah and Albouy (2001), p. 430.) Parameterisation of nonosculation through a non-Lagrange constraint was offered in Efroimsky (2002a,b).

Similarly to orbital dynamics, in attitude dynamics, a complex spin can be presented as a sequence of instantaneous configurations borrowed from a family of some "simple rotations" (Efroimsky 2004). It is convenient to employ in this role the motions exhibited by an undeformable free top experiencing no torques.[3] Each such undisturbed "simple motion" will be a trajectory on the three-dimensional manifold of the Euler angles (Synge and Griffith 1959). For the lack of a better term, we shall call these unperturbed motions "Eulerian cones,"

---

[3] Here one opportunity will be to utilise in the role of "simple" motions the noncircular Eulerian cones described by the actual triaxial top, when it is unforced. Another opportunity will be to use, as "simple" motions, the circular Eulerian cones described by a dynamically symmetrical top (and to treat its actual triaxiality as another perturbation). The main result of our paper will be invariant under this choice.

implying that the loci of the rotational axis, which correspond to each such nonperturbed spin state, make closed cones (circular, for an axially symmetrical rotator; and elliptic for a triaxial one). Then, to implement a perturbed motion, we shall have to go from one Eulerian cone to another, just as in Figs. 1 and 2 we go from one Keplerian ellipse to another. Hence, similar to those pictures, a smooth "walk" over the instantaneous Eulerian cones may be osculating or nonosculating.

The torques, as well as the actual triaxiality of the top and the noninertial nature of the reference frame, will then act as perturbations causing this "walk." Perturbations of the latter two types depend not only upon the rotator's orientation but also upon its angular velocity.[4]

2.2 Delaunay and Andoyer

In orbital dynamics, we can express the Lagrangian of the reduced two-body problem via the spherical coordinates $q_j = \{r, \varphi, \theta\}$, then derive their conjugated momenta p_j and the Hamiltonian $\mathcal{H}(q, p)$, and then carry out the Hamilton–Jacobi procedure (Plummer 1918), to arrive at the Delaunay variables

$$\{Q_1, Q_2, Q_3; P_1, P_2, P_3\} \equiv \{L, G, H; l_o, g, h\}$$
$$= \{\sqrt{\mu a}, \sqrt{\mu a \left(1 - e^2\right)}, \sqrt{\mu a \left(1 - e^2\right)} \cos i; -M_o, -\omega, -\Omega\}, \tag{1}$$

where μ denotes the reduced mass.

Similarly, in rotational dynamics one can define a state of a spinning top by the three Euler angles $q_j = \{\varphi, \theta, \psi\}$ and their canonical momenta $p_j = \{p_\varphi, p_\theta, p_\psi\}$; and then carry out a canonical transformation to the Andoyer elements[5] $\{l, g, h; L, G, H\}$. By definition, the element G is the magnitude of the angular-momentum vector, L is the projection of the angular-momentum vector on the principal axis $\hat{\mathbf{b}}_3$ of the body, while H is the projection of the angular-momentum vector on the $\hat{\mathbf{s}}_3$ axis of the inertial coordinate system. The variable h conjugate to H is the angle from the inertial reference longitude to the ascending node of the invariable plane (the one perpendicular to the angular momentum). The variable g conjugate to G is the angle from the ascending node of the invariable plane on the reference plane to the ascending node of the equator on the invariable plane. Finally, the variable conjugate to L is the angle l from the ascending node of the equator on the invariable plane to the the $\hat{\mathbf{b}}_1$ body axis. Two auxiliary quantities defined through

$$\cos I = \frac{H}{G}, \quad \cos J = \frac{L}{G},$$

have obvious meaning: I is the angle between the angular-momentum vector and the $\hat{\mathbf{s}}_3$ space axis, while J is the angle between the angular-momentum vector and the $\hat{\mathbf{b}}_3$ principal axis of the body, as depicted on Fig. 3.

Andoyer (1923) introduced his variables in a manner different from canonical constants: while his variables G, H, h are constants (for a free triaxial rotator), the other three, L, l, g,

---

[4] When we study the Earth rotation relative to the precessing plane of the Earth orbit about the Sun, the frame precession gives birth to a fictitious torque (sometimes called "inertial torque") that depends upon the Earth's angular velocity.

[5] In attitude dynamics, the Andoyer elements l, g, h play the role of coordinates, while L, G, H are their conjugate momenta. In the orbital case, the Delaunay variables L, G, H play the role of coordinates, while l, g, h defined as in (2.2) act as momenta. Needless to say, this is merely a matter of convention. (See formulae (9.31–9.32) in Goldstein 1981.) For example in some textbooks on orbital mechanics the Hamiltonian perturbation is deliberately introduced with an opposite sign, while the Delaunay elements l, g, h, too, are defined with signs opposite to given in (2.2). Under such a convention, the Delaunay elements l, g, h become coordinates, while L, G, H act as momenta.
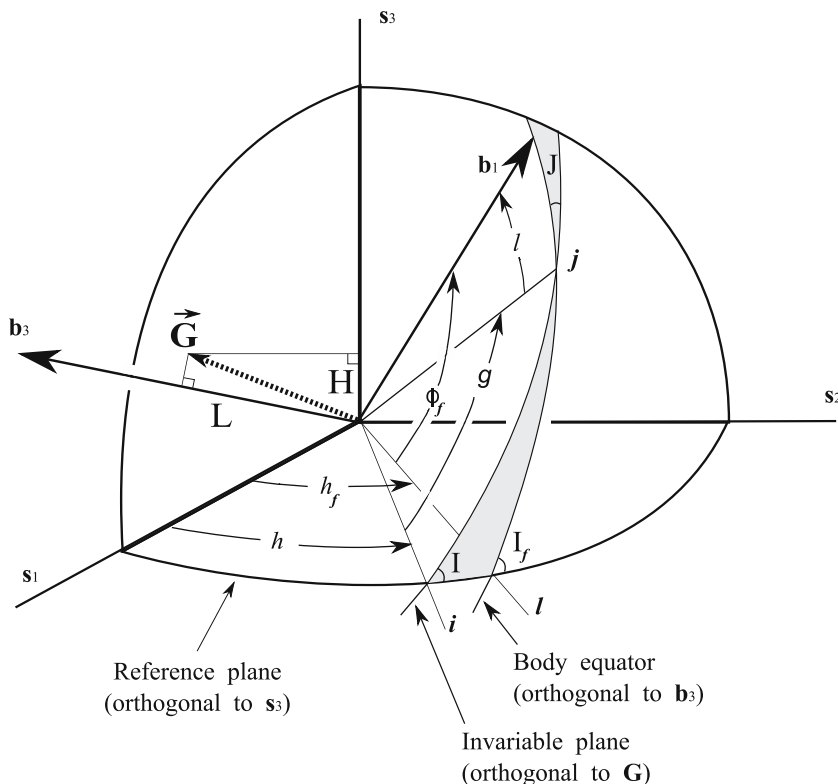
Fig. 3 A reference coordinate system (inertial or precessing) is constituted by axes $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3$. A body-fixed frame is defined by the principal axes $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$. The third frame is constituted by the angular-momentum vector $\vec{\mathbf{G}}$ and a plane orthogonal thereto (the so-called invariable plane). The lines of nodes are denoted with $\mathbf{i}, \mathbf{l}, \mathbf{j}$. The attitude of the body relative to the reference frame is given by the Euler angles h_f, I_f, ϕ_f. The orientation of the invariable plane with respect to the reference frame is determined by the angles h and I. The inclination I is equal to the angle that the angular-momentum vector $\vec{\mathbf{G}}$ makes with the reference axis $\mathbf{s}_3$. The angle J between the invariable plane and the body equator coincides with the angle that $\vec{\mathbf{G}}$ makes with the major-inertia axis $\mathbf{b}_3$ of the body. The projections of the angular momentum toward the reference axis $\mathbf{s}_3$ and the body axis $\mathbf{b}_3$ are $H = G \cos I$ and $L = G \cos J$

do evolve in time, because the Andoyer Hamiltonian of a free top

$$\mathcal{H}(g, h, l, G, H, L) = \frac{1}{2} \left(\frac{\sin^2 l}{A} + \frac{\cos^2 l}{B} \right) \left(G^2 - L^2 \right) + \frac{L^2}{2C}$$

is a nonvanishing function of the variables l, L and G. (Notations A, B, C stand for the inertia matrix' principal values that are assumed, without loss of generality, to obey the inequality $A \leq B \leq C$.) So, to make our analogy complete, we may carry out one more canonical transformation, from the regular Andoyer set $\{l, g, h, L, G, H\}$ to the "modified Andoyer set" $\{l_o, g_o, h; L_o, G, H\}$, where L_o, l_o, g_o are the initial values of L, l, g. The modified set consists only of constants of integration, wherefore the appropriate Hamiltonian becomes nil.[6] Therefore, the modified Andoyer set of variables is analogous to the Delaunay set with

---

[6] We shall not write down the explicit form of this transformation, because it is sufficient for us to know that it is canonical. This follows from the group property of canonical transformations and from the fact

$l_o = -M_o$, while the regular Andoyer elements are analogous to the Delaunay elements with $l = -M$ used instead of $l_o = -M_o$. We would stress that, in analogy with the orbital case, the variables h, G, H are constants (and $h_o = h$, $G_o = G$, $H_o = H$) only in the unperturbed, free-spin, case.

To summarise this section, in both cases we start out with

$$\dot{q} = \frac{\partial \mathcal{H}^{(o)}}{\partial p}, \quad \dot{p} = -\frac{\partial \mathcal{H}^{(o)}}{\partial q}, \tag{2}$$

q and p being the coordinates and their conjugated momenta, in the orbital case, or the Euler angles and their momenta, in the rotation case. Then we switch, via a canonical transformation

$$q = f(Q, P, t),$$
$$p = \chi(Q, P, t) \tag{3}$$

to

$$\dot{Q} = \frac{\partial \mathcal{H}^*}{\partial P} = 0, \quad \dot{P} = -\frac{\partial \mathcal{H}^*}{\partial Q} = 0, \quad \mathcal{H}^* = 0, \tag{4}$$

where Q and P denote the set of Delaunay elements, in the orbital case, or the (modified, as explained above) Andoyer set $\{l_o, g_o, h; L_o, G, H\}$, in the case of rigid-body rotation.

This scheme relies on the fact that, for an unperturbed Keplerian orbit (and, similarly, for an undisturbed Eulerian cone), its six-constant parameterisation may be chosen so that:

1. the parameters are constants and, at the same time, are canonical variables $\{Q, P\}$ with a zero Hamiltonian: $\mathcal{H}^*(Q, P) = 0$;
2. for constant Q and P, the transformation equations (3) are mathematically equivalent to the dynamical equations (2).

In practice, this scheme is implemented via the Hamilton-Jacobi procedure.

2.3 Canonical perturbation theory: canonicity versus osculation

Under perturbation, the "constants" Q, P begin to evolve so that, after their insertion into

$$q = f(Q(t), P(t), t),$$
$$p = \chi(Q(t), P(t), t) \tag{5}$$

(f and χ being the same functions as in (3)), the resulting motion obeys the disturbed equations

$$\dot{q} = \frac{\partial \left(\mathcal{H}^{(o)} + \Delta\mathcal{H} \right)}{\partial p}, \quad \dot{p} = -\frac{\partial \left(\mathcal{H}^{(o)} + \Delta\mathcal{H} \right)}{\partial q}. \tag{6}$$

Footnote 6 continued

that the transformations from $\{l, g, h, L, G, H\}$ to $\{\varphi, \theta, \psi, p_\varphi, p_\theta, p_\psi\}$ and from $\{l_o, g_o, h; L_o, G, H\}$ to $\{l, g, h, L, G, H\}$ are canonical. The latter transformation is canonical, for it is simply the time evolution. This canonical transition from Andoyer-type variables to their initial values is not new—see Fukushima and Ishizaki (1994). Historically, the first set of rotational elements was constituted by constants (Richelot 1850). Serret (1866) found the generating function of a canonical transformation from $\{\varphi, \theta, \psi, p_\varphi, p_\theta, p_\psi\}$ to that set. His development was polished by Radau (1869) and Tisserand (1889). The Serret–Richelot set consisted of the following constants: $\{g_o, h, -t_o; G, H, T_{\text{kin}}\}$, where h, G and H coincide with the appropriate Andoyer elements, T_{kin} is the rotational kinetic energy, t_o is the initial moment of time, and g_o is the initial value of the Andoyer element g.

We also want our "constants" Q and P to remain canonical and to obey

$$\dot{Q} = \frac{\partial \, (\mathcal{H}^* + \Delta \mathcal{H}^*)}{\partial \, P}, \quad \dot{P} = -\frac{\partial \, (\mathcal{H}^* + \Delta \mathcal{H}^*)}{\partial \, Q}, \tag{7}$$

where

$$\mathcal{H}^* = 0 \text{ and } \Delta \mathcal{H}^* \, (Q, \, Pt) = \Delta \mathcal{H} \, (q(Q, \, P, \, t), \, p(Q, \, P, \, t), \, t) \,. \tag{8}$$

Above all, we wish that the perturbed "constants" $C_j \equiv Q_1, \, Q_2, \, Q_3, \, P_1, \, P_2, \, P_3$ (the Delaunay elements, in the orbital case, or the modified Andoyer elements, in the rotation case) remain osculating. This means that the perturbed velocity will be expressed by the same function of $C_j(t)$ and t as the unperturbed one used to. Let us check to what extent this optimism is justified. The perturbed velocity reads

$$\dot{q} = g + \Phi, \tag{9}$$

where

$$g(C(t), t) \equiv \frac{\partial q(C(t), t)}{\partial t} \tag{10}$$

is the functional expression for the unperturbed velocity; and

$$\Phi(C(t), t) \equiv \sum_{j=1}^{6} \frac{\partial q(C(t), t)}{\partial C_j} \dot{C}_j(t) \tag{11}$$

is the convective term. Since we chose the "constants" C_j to make canonical pairs (Q, P) obeying (7–8) with vanishing $\mathcal{H}^*$, then insertion of (7) into (11) will result in

$$\Phi = \sum_{n=1}^{3} \frac{\partial q}{\partial Q_n} \dot{Q}_n(t) + \sum_{n=1}^{3} \frac{\partial q}{\partial P_n} \dot{P}_n(t) = \frac{\partial \Delta \mathcal{H}(q, \, p)}{\partial p}. \tag{12}$$

We see that in some situations the canonicity requirement is incompatible with osculation.[7] To be specific, under a momentum-dependent perturbation we still can use ansatz (5) for calculation of the coordinates and momenta, but cannot impose the osculation condition $\Phi = 0$ (i.e., we cannot use $\dot{q} = g$ for calculating the velocities). Instead, we must use (9) with the substitution (12). This generic rule applied both to orbital and rotational motions. Its application to the orbital case is illustrated by Fig. 2. There, the constants $C_j = (Q_n, P_n)$ parameterise instantaneous ellipses which, for nonzero Φ, are *not* tangent to the trajectory. In orbital mechanics, the variables preserving canonicity at the cost of osculation are called "contact elements" (term coined by Victor Brumberg). The osculating and contact variables coincide when the disturbance is velocity-independent. Otherwise, they differ already in the first order of the time-dependent perturbation (Efroimsky and Goldreich 2003, 2004). Luckily, in some situations, their secular parts differ only in the second order (Efroimsky 2005), a fortunate circumstance anticipated by Goldreich (1965), who came across these elements in a totally different context unrelated to canonicity.

The case of rotational motion will parallel the theory of orbits. Now, instead of the instantaneous Keplerian conics, one will deal with instantaneous Eulerian cones (i.e., with the loci of the rotational axis, corresponding to nonperturbed spin states). Indeed, the situation of

---

[7] For the first time, this observation was made in Efroimsky and Goldreich (2003).

an axially symmetric unsupported top at each instant of time is fully defined by the three
Euler angles $q_n = \phi, \theta, \psi$ and their time derivatives $\dot{q}_n = \dot{\phi}, \dot{\theta}, \dot{\psi}$. The evolution of these six
quantities is governed by three dynamical equations of the second order (the three projections
of $d\vec{\mathbf{L}}/dt = \vec{\tau}$, where $\vec{\mathbf{L}}$ is the angular momentum and $\vec{\tau}$ is the torque) and, therefore, this
evolution will depend upon the time and the six integration constants:

$$
\begin{aligned}
q_n &= f_n \, (C_1, \ldots, C_6, t) \,, \\
\dot{q}_n &= \mathrm{g}_n \, (C_1, \ldots, C_6, t) \,,
\end{aligned}
\tag{13}
$$

where the functions g_n and f_n are interconnected via $\mathrm{g}_n \equiv \partial f_n / \partial t$, for $n = \psi, \theta, \phi$.

Under disturbance, the motion will be altered:

$$
\begin{aligned}
q_n &= f_n \, (C_1(t), \ldots, C_6(t), t) \,, \\
\dot{q}_n &= \mathrm{g}_n \, (C_1(t), \ldots, C_6(t), t) + \Phi_n \, (C_1(t), \ldots, C_6(t), t) \,,
\end{aligned}
\tag{14}
$$

where

$$
\Phi_n \, (C_1(t), \ldots, C_6(t), t) \equiv \sum_{j=1}^{6} \frac{\partial f_n}{\partial C_j} \dot{C}_j.
\tag{15}
$$

If we want the "constants" C_j to constitute canonical pairs (Q, P) obeying (7–8), then
insertion of (7) into (15) will result in

$$
\Phi_n \, (C_1(t), \ldots, C_6(t), t) \equiv \sum \frac{\partial f_n}{\partial Q} \dot{Q} + \sum \frac{\partial f_n}{\partial P} \dot{P} = \frac{\partial \Delta \mathcal{H}(q, p)}{\partial p_n},
\tag{16}
$$

so that the canonicity requirement (7–8) violates the gauge freedom in a non-Lagrange
fashion.

To draw this subsection to a close, let us sum up two facts. First, no matter what the
Hamiltonian perturbation is to be, the Delaunay (in the orbital case) or the modified Andoyer
(in the attitude case) variables Q, P always remain canonical. They do so simply because they
are *a priori* defined to be canonical—see Eqs. 4 and 7–8 above. Second, as we have seen from
(15–16), the osculating character of the Q, P variables is lost under momentum-dependent
perturbations of the Hamiltonian.[8]

2.4 From the modified Andoyer elements to the regular ones

So far our description of perturbed spin, (13–16), has merely been a particular case of the
general development (5–12). The sole difference was that the role of canonicallyconjugat-
ed integration constants $C = (Q, P)$ in (13–16) should be played not by the Delaunay
variables (as in the orbital case) but by some rotational elements—like, for example, the
Richelot–Serret variables (see the footnote in Sect. 2.2 above) or by the modified Andoyer
set $(l_o, g_o, h; L_o, G, H)$ consisting of the initial values of the regular Andoyer elements.
The developments conventionally used in the theory of Earth rotation, as well as in space-
craft attitude engineering, are almost always set out in terms of the regular Andoyer ele-
ments, not in terms of their initial values (the paper by Fukushima and Ishizaki (1994)
being a unique exception). Fortunately, all our gadgetry, developed above for the modified

---

[8] It is possible, of course, to choose the other way and preserve osculation at the cost of canonicity. In the
orbital case, one should simply set $\vec{\Phi} = 0$ in Eqs. 52–57 of Efroimsky (2006). In the attitude case, though,
this will be a more cumbersome construction, never implemented in the literature hitherto.

Andoyer set, stays applicable for the regular set. To prove this, let us consider the unperturbed parameterisation of the Euler angles $q_n = (\phi, \theta, \psi)$ via the regular Andoyer elements $A_j = (l, g, h; L, G, H)$:

$$q_n = f_n (A_1(C, t), \ldots, A_6(C, t)), \tag{17}$$

each element A_i being a function of time and of the initial values $C_j = (l_o, g_o, h; L_o, G, H)$. When a perturbation gets turned on, the parameterisation (17) stays, while the time evolution of the elements A_i changes: beside the standard time-dependence inherent in the free-spin Andoyer elements, the perturbed elements acquire an extra time-dependence through the evolution of their initial values.[9] Then the time evolution of an Euler angle $q_n = (\varphi, \theta, \psi)$ will be given by a sum of two items: (1) the angle's unperturbed dependence upon time and time-dependent elements; and (2) an appropriate addition Φ_n that arises from a perturbation-caused alteration of the elements' dependence upon the time:

$$\dot{q}_n = g_n + \Phi_n. \tag{18}$$

The unperturbed part is

$$g_n = \sum_{i=1}^{6} \frac{\partial f_n}{\partial A_i} \left(\frac{\partial A_i}{\partial t} \right)_C, \tag{19}$$

while the convective term is given by

$$\begin{aligned}
\Phi_n &= \sum_{i=1}^{6} \sum_{j=1}^{6} \left(\frac{\partial f_n}{\partial A_i} \right)_t \left(\frac{\partial A_i}{\partial C_j} \right)_t \dot{C}_j = \sum_{j=1}^{6} \left(\frac{\partial f_n}{\partial C_j} \right)_t \dot{C}_j \\
&= \sum_{j=1}^{3} \left(\frac{\partial f_n}{\partial Q_j} \right)_t \dot{Q}_j + \sum_{j=1}^{3} \left(\frac{\partial f_n}{\partial P_j} \right)_t \dot{P}_j = \frac{\partial \Delta \mathcal{H}(q, p)}{\partial p_n},
\end{aligned} \tag{20}$$

where the set C_j is split into canonical coordinates and momenta like this: $Q_j = (l_o, g_o, h)$ and $P_j = (L_o, G, H)$. In the case of free spin they obey the Hamilton equations with a vanishing Hamiltonian and, therefore, are all constants. In the case of disturbed spin, their evolution is governed by (7–8), substitution whereof in (20) will once again take us to (16). This means that the nonosculation-caused convective corrections to the velocities stay the same, no matter whether we parameterise the Euler angles through the modified Andoyer elements (variable constants) or through the regular Andoyer elements. This invariance will become obvious if, once again, we consider the analogy with orbital mechanics: on Fig. 1, the correction $\vec{\Phi}$ is independent of how we choose to parameterise the nonosculating instantaneous ellipse—through the Delaunay set with M_o or through the one containing M.

This consideration yields the following consequences:

---

[9] This is fully analogous to the transition from the unperturbed mean longitude,

$$M(t) = M_o + n(t - t_o), \quad \text{with} \quad M_o, n, t_o = \text{const},$$

to the perturbed one,

$$M(t) = M_o(t) + \int_{t_o}^{t} n(t')dt', \quad \text{with} \quad t_o = \text{const},$$

in orbital dynamics.

(a) Under momentum-dependent perturbations, calculation of the angular velocities via the elements must be performed not through the second equation of (13) but through the second equation of (13), with (16) substituted therein. The convective term given by (16) is nonzero when the perturbation is angular-velocity-dependent. In other words, under such type of perturbations, the canonicity condition imposed upon the Richelot–Serret or the Andoyer elements is incompatible with osculation. An example of such perturbation shows itself in the theory of planetary rotation, when we switch to a coordinate system associated with the orbit plane. Precession of this plane makes the frame noninertial, and the appropriate Lagrangian perturbation depends upon the planet's angular velocity. The corresponding Hamiltonian perturbation (denoted in the Kinoshita–Souchay theory by E) comes out momentum-dependent. In this theory the Andoyer elements are introduced in the precessing frame, and since the precession-caused perturbation is momentum-dependent, these elements come out nonosculating. For this reason, their substitution into the undisturbed expressions (2.6) and (6.26–6.27) in Kinoshita (1977) will not render the angular velocity relative to the precessing frame wherein the elements were introduced. To furnish the angular velocity relative to that frame, these expressions must be amended with the appropriate convective terms.

(b) The above circumstance, instead of being a flaw of the Kinoshita–Souchay theory, turns out to be its strong point. It can be shown that Kinoshita's undisturbed expressions for the angular velocity via the elements keep rendering the *inertial* angular velocity, even when the elements defined in a precessing frame are plugged therein. Briefly speaking, we first introduce the Andoyer elements in an unperturbed setting (inertial frame) and write down the expressions, via these elements, for the Euler angles and velocities relative to the inertial frame. Then we introduce a momentum-dependent perturbation, i.e., switch to a precessing frame, and in that frame we introduce the Andoyer elements. Insertion thereof into the unperturbed expressions for the Euler angles and angular velocities gives us the Euler angles relative to the precessing frame and (due to the nonosculating nature of the elements) the angular velocity relative to the *inertial* frame, not to the precessing one.[10] Proof of this fact is presented in Appendix 1.3.

This fact should not be regarded as a disadvantage of the Kinoshita–Souchay theory, because in some situations it is the inertial, not the relative, angular velocity that is measured (Schreiber et al. 2004; Petrov 2007). Under these circumstances, the Kinoshita–Souchay expressions for the angular velocity should be employed (as long as they are correctly identified as the formulae for the *inertial* angular velocity).

---

[10] This mishap is an example of osculation loss. We introduce the elements in a certain frame (the precessing frame of the orbit), plug them into the unperturbed expressions for the Euler angles and for the angular velocities, and here comes the result: while we obtain the correct values of the Euler angles relative to the said frame, we do *not* get the right values for the angular velocity relative to that frame. (Instead, our formulae return the values of the angular velocity relative to another, inertial, frame.) This happens because the disturbance, associated with a transition to the precessing frame, depends not only upon the Earth's orientation but also upon its angular velocity. Or, stated alternatively, because the appropriate Hamiltonian variation $\Delta\mathcal{H}$ depends upon the momenta p canonically conjugated to the Euler angles q:

$$\Phi \equiv \frac{\partial \Delta\mathcal{H}}{\partial p} \neq 0.$$

3 The angular velocity relative to the precessing frame

In the theory of Earth rotation, three angular velocities emerge:

$\vec{\omega}^{(\text{rel})}$ ≡ the relative angular velocity,
     i.e., the body's angular velocity relative to a precessing orbital frame;

$\vec{\mu}$ ≡ the precession rate of the orbital frame with respect to an inertial one;

$\vec{\omega}^{(\text{inert})}$ ≡ the inertial angular velocity,
     i.e., the body's angular velocity with respect to the inertial frame.

Evidently, the latter is the sum of the two former ones: $\vec{\omega}^{(\text{inert})} = \vec{\omega}^{(\text{rel})} + \vec{\mu}$.

If some day we develop an experimental technique for measuring the Earth's angular velocity relative to the precessing plane of its orbit, we shall have to compare the observations with the theoretical predictions for the directional angles of this, *relative*, angular velocity $\vec{\omega}^{(\text{rel})}$.

The Kinoshita (1977) theory was created with intention to furnish the precessing-frame-related directional angles[11] of the Earth figure (formulae (2.3) and (6.24–6.25) in Kinoshita's paper). This theory also provides precessing-frame-related directional angles of the Earth's angular-velocity vector (formulae (2.6) and (6.26–6.27) in *Ibid.*). We prove in Appendix 1.3 below that, contrary to the expectations, the latter expressions render the directional angles not of the relative but of the *inertial* angular velocity $\vec{\omega}^{(\text{inert})}$:

$$I_r^{(\text{inert})} = I + J \left(1 - \frac{C}{2A} - \frac{C}{2B} \right) [\cos g - e \cos (2l + g)], \qquad (21)$$

and

$$h_r^{(\text{inert})} = h + \frac{J}{\sin I} \left(1 - \frac{C}{2A} - \frac{C}{2B} \right) [\sin g - e \sin (2l + g)]. \qquad (22)$$

where the angles I and J are as on Fig. 3, while e is introduced as a measure of triaxiality of the rotator:[12]

$$e \equiv \frac{[(1/B) - (1/A)]/2}{(1/C) - [(1/A) + (1/B)]/2} \qquad (23)$$

A, B, C being the principal moments of inertia.[13]

This is a very nontrivial and counterintuitive fact. On introducing the Andoyer variables in the precessing frame of orbit, we plug them into the standard expressions for the orientation angles and the angular velocity. Doing so, we naturally expect to obtain the orientation and the spin rate relative to that precessing frame. We indeed get the body orientation relative to that frame, but the rendered angular velocity turns out to be *not* the one relative to the precessing frame wherein the Andoyer elements were introduced. Instead, the standard formulae give us the angular velocity relative to some other frame, the inertial one (as if we had used the

---

[11] Here and hereafter the term "directional angles" will stand for the longitude of the node and the inclination of the plane perpendicular to the Earth figure. An analogous meaning is understood for the directional angles of the angular velocity.

[12] For the Earth, $J \sim 10^{-6}$ rad, which justifies the common approximation to write all formulae up to the first order in J (Kinoshita 1977). The value of the triaxiality parameter is: $e = 3.3646441 \times 10^{-3}$ (Escapa et al. 2002).

[13] The Earth is assumed to be rigid, and its body axes are chosen to diagonalise its inertia matrix.

Andoyer variables defined in the inertial frame). This is an interesting (and still underappreciated by mathematicians) internal symmetry instilled into the Andoyer construction: we can go through a continuum of Andoyer sets (each set introduced in a different precessing frame), but their substitution into the standard formulae for the angular velocity will always return the angular velocity relative to the inertial frame.

A proof of this fact begins with a study of the physical meaning of the Andoyer elements introduced in a precessing frame (presented in Appendix A.1.1). Completion of the proof demands a sequence of calculations so laborious that we chose to put them into the Appendix (see Appendices A.1.2–A.1.3). This entire situation remarkably parallels a similar episode from the theory of Delaunay elements in orbital dynamics (see the end of Appendix A.1.3).

Now, what if we want to know the angular velocity relative to the precessing frame, i.e., $\vec{\omega}^{(\text{rel})}$? The precessing-frame-related directional angles of this angular velocity will look as

$$I_r^{(\text{rel})} \; = \; I_r^{(\text{inert})} + I_r^{(\Phi)}, \tag{24}$$

and

$$h_r^{(\text{rel})} = h_r^{(\text{inert})} + h_r^{(\Phi)}, \tag{25}$$

the extra terms $I_r^{(\Phi)}$; $h_r^{(\Phi)}$ emerging because, as explained above, one has to add the convective term Φ to the unperturbed velocity g, in order to obtain the full velocity $\dot{q}$ under disturbance. Here q stands for the three Eulerian angles[14] $q_n = \{h_f, I_f, \phi_f\}$ defining the the orientation of the principal axes of the Earth, relative to the precessing frame, so $\dot{q}_n$ will signify time derivatives of Euler angles relative to this precessing frame. The convective terms, entering the expressions for $\dot{q}_n = \{\dot{h}_f, \dot{I}_f, \dot{\phi}_f\}$, can be calculated using formula (16)—see the Appendices 3 and 4 below. The ensuing corrections to the Euler angles determining the orientation of the instantaneous spin axis will look as

$$I_r^{(\Phi)} \; = \; -\dot{\pi}_1 \frac{C}{L} \cos I \sin(h - \Pi_1) + \dot{\Pi}_1 \frac{C}{L} [\sin \pi_1 \cos I \cos(h - \Pi_1)$$
$$+ \cos \pi_1 \sin I - \sin I] + O\left(J^2\right) + O(J\Phi/\omega) + O((\Phi/\omega)^2) \tag{26}$$

and

$$h_r^{(\Phi)} = -\dot{\pi}_1 \frac{C}{L} \frac{\cos(h - \Pi_1)}{\sin I} - \dot{\Pi}_1 \frac{C}{L} \frac{\sin \pi_1 \sin(h - \Pi_1)}{\sin I}$$
$$+ O(J^2) + O((\Phi/\omega)^2) + O(J\Phi/\omega). \tag{27}$$

These corrections depend upon two angles that define the orientation of the precessing orbit with respect to an inertial frame—the inclination π_1 and the node Π_1.

---

[14] Be mindful that in the physics and engineering literature the Euler angles are traditionally denoted with (ϕ, θ, ψ). In the literature on the Earth rotation, very often the inverse convention, (ψ, θ, ϕ), is employed. In the Kinoshita–Souchay theory, these angles are denoted with (h, I, ϕ). The angles defining orientation of the Earth's figure are accompanied with the subscript f and are termed as: (h_f, I_f, ϕ_f). The directional angles of the Earth's angular-velocity vector are equipped with subscript r. For the relative and the inertial angular velocities the angles are denoted with $(h_r^{(\text{rel})}, I_r^{(\text{rel})}, \phi_r^{(\text{rel})})$ and $(h_r^{(\text{inert})}, I_r^{(\text{inert})}, \phi_r^{(\text{inert})})$, accordingly.

Let us make rough numerical estimates for the case of the rigid Earth. Putting together (21), (24), and (26), we see that in the resulting expression for $I_r^{(rel)}$

$$
\begin{aligned}
I_r^{(rel)} = I + J &\left(1 - \frac{C}{2A} - \frac{C}{2B}\right)[\cos g - e\cos(2l+g)] \\
&-\dot{\pi}_1 \frac{C}{L}\cos I\sin(h - \Pi_1) + \dot{\Pi}_1 \frac{C}{L}[\sin\pi_1\cos I\cos(h - \Pi_1) \\
&+ \cos\pi_1\sin I - \sin I] + O(J^2) + O(J\Phi/\omega) + O((\Phi/\omega)^2) \quad (28)
\end{aligned}
$$

we have the leading term, I, and three additions—of order $J \sim 10^{-6}$, of order $Je \sim 10^{-9}$, and of order[15] $\Phi/\omega \sim \dot{\pi}_1/\omega \sim 10^{-9}$. We see that *the nonosculation-caused convective terms* Φ *provide an effect on the spin-axis orientation, which is of the same order as the Je term stemming from triaxiality.*[16] As $1\,rad \approx 0.2 \times 10^{6\prime\prime}$ and $J \sim 10^{-6}$, then the J term brings into I_r a contribution of an arcsecond order, while the Φ and Je terms give corrections of order milliarcseconds. We also see that the terms of order $J\Phi/\omega$ and those of $(\Phi/\omega)^2$ are much less than one percent of a microarcsecond and may be neglected. Numerical estimates for the expression

$$
\begin{aligned}
h_r^{(rel)} = h + \frac{J}{\sin I}&\left(1 - \frac{C}{2A} - \frac{C}{2B}\right)[\sin g - e\sin(2l+g)] \\
&-\dot{\pi}_1 \frac{C}{L}\frac{\cos(h - \Pi_1)}{\sin I} - \dot{\Pi}_1 \frac{C}{L}\frac{\sin\pi_1\sin(h - \Pi_1)}{\sin I} + O(J^2) \\
&+ O((\Phi/\omega)^2) + O(J\Phi/\omega) \quad (29)
\end{aligned}
$$

will be similar.

Formulae (26–27) constitute the main result of this paper.[17] In Appendices 2–4 we present their derivation based on formulae (18) and (20). It would be important to note that the resulting corrections acquire the form (25–27) provided the coordinate system co-precessing with the orbit is chosen as in Kinoshita (1977), i.e., by three consecutive Euler rotations $\hat{\mathbf{R}}_3(-\Pi_1)\hat{\mathbf{R}}_N(\pi_1)\hat{\mathbf{R}}_Z(\Pi_1)$, letter Z standing for an inertial axis orthogonal to the ecliptic of epoch, N denoting the line of nodes, and 3 being a precessing axis perpendicular to the ecliptic of date—see Appendix A.2.2.3. Under an alternative choice of axes within the co-precessing frame, expressions for $I_r^{(rel)}$ and $h_r^{(rel)}$ will look differently. For example, a transition carried out by only two Euler rotations, $\hat{\mathbf{R}}_N(\pi_1)\hat{\mathbf{R}}_Z(\Pi_1)$, as in Appendix A.2.2.2, will yield expression (114) instead of (28), and (123) instead of (27).

In principle, (25–27) might as well be derived by purely geometrical means, i.e., from the formula $\vec{\omega}^{(inert)} = \vec{\omega}^{(rel)} + \vec{\mu}$. We however chose the method based on (18) and (20), because this method is fundamental and applicable to *any kind* of momentum-dependent perturbations of the Hamiltonian—for example, to the perturbations caused by deviations from rigidity, as studied by Getino and Ferrándiz (1994). A similar situation will emerge in the (yet to be built) relativistic theory of the Earth rotation.

---

[15] This estimate ensues from the trivial observation that $C/L \approx \omega^{-1}$. Regarding the numbers: according to Lieske et al. (1977) and Seidelmann (1992), $\dot{\pi}_1 \sim 47\prime\prime$/century, while $\dot{\Pi}_1 \sim -870\prime\prime$/century $\approx -2.4 \times 10^{-3}$deg/yr. On the other hand, $\omega \sim 360$ deg/day $\sim 1.3 \times 10^5$deg/yr whence $\Phi/\omega \sim \dot{\pi}_1/\omega \sim 10^{-9}$. (We could as well have used the IERS value of $\omega \approx 7.3 \times 10^{-5} rad/s \sim 1.3 \times 10^7$deg/century $\sim 4.7 \times 10^{10\prime\prime}$/century.)

[16] The nutational spectra of these two contributions are, however, quite different (secular vs. periodic).

[17] Through the medium of Eqs. 62–64 it is also possible to express these corrections via the Euler set, instead of the Andoyer variables.

4 Conclusions

In this article we explained that the unperturbed spin states ("Eulerian cones") play in the attitude dynamics the same role as the unperturbed two-body orbits ("Keplerian conics") play in the orbital mechanics. Just as the orbital elements parameterising Keplerian conics, the rotational elements parameterising Eulerian cones may be either osculating or nonosculating. If the perturbation depends upon the velocity (in the orbital case) or upon the angular velocity (in the attitude case), the condition of osculation is incompatible with the condition of canonicity. In these situations the standard equations furnish the Delaunay (in the orbital case) or Andoyer (in the attitude case) elements, which are not osculating—circumstance important when the elements are employed for calculation of the velocity or angular velocity. The functional form of the expression for a velocity or an angular velocity through elements depends upon whether these elements are osculating or not.

A remarkable peculiarity is shared by the Delaunay and Andoyer elements. Suppose the perturbation is caused by a transition to a precessing frame of reference, and the elements are introduced in this noninertial frame. Their substitution into the unperturbed expressions for the Cartesian coordinates (or the Euler angles) will render the right position (or the attitude) relative to the precessing frame wherein these elements were defined. Now, suppose that we impose on our elements the condition of canonicity. Since the frame-precession-caused perturbation is momentum-dependent, the canonicity condition is incompatible with the osculation one. Hence, when our elements are inserted into the unperturbed expressions for the velocity or angular velocity, they will NOT return the velocity with respect to the precessing frame. It turns out, though, that they will render the velocity relative to the inertial frame. While for the orbital case this was proven in Efroimsky and Goldreich (2003, 2004), in the current paper we proved this fact also for the attitude case.

This has ramifications for the Kinoshita–Souchay theory of the Earth rotation. In this theory, the Andoyer elements are defined in a precessing frame of the Earth orbit. In Kinoshita (1977) these elements were *ab initio* canonical—simply because Kinoshita obtained them via a canonical transformation (see Sect. 3 of his work). As demonstrated in Sects. 2.3–2.4 of our paper, the by-default-imposed canonicity condition made the elements nonosculating. Insertion of such elements into the unperturbed equations for the angular velocity (formulae (2.6) and (6.26–6.27) in Kinoshita 1977) does not yield the angular velocity relative to the frame wherein the elements were defined (the precessing frame). Rather, the equations will still furnish the angular velocity relative to the inertial frame of reference. This way of oscu-lation loss might be a flaw of the Kinoshita–Souchay theory, had we expected it to render the angular velocity with respect to a precessing frame. In reality, the osculation loss is an advantage of the theory, because the presently available experimental technique (Schreiber et al. 2004; Petrov 2007) provides for the measurement of the angular velocity relative to the inertial frame–the velocity furnished by the Kinoshita–Souchay theory.

In the final section we provide expressions (26–27) for the body-frame-related directional angles of the planet's angular velocity relative to a frame coprecessing with the planet's orbit. The method wherewith we calculate these angles is general and applicable to to *any kind* of momentum-dependent perturbations of the Hamiltonian—for example, to the perturbations caused by deviations from rigidity.

Appendix 1: The Andoyer variables introduced in a precessing frame

A1.1 Formalism

Let us consider an unsupported rigid body whose spin is to be studied in a coordinate system, which itself is precessing relative to some inertial frame. The said system is assumed to precess at a rate $\vec{\mu}$ so the kinetic energy of rotation, in the inertial frame, is given by

$$
\begin{aligned}
T_{\text{kin}} &= \frac{1}{2} \vec{\omega}^{(\text{inert})T} \mathbb{I} \vec{\omega}^{(\text{inert})} = \frac{1}{2} \left(\vec{\omega}^{(\text{rel})} + \vec{\mu} \right)^{T} \mathbb{I} \left(\vec{\omega}^{(\text{rel})} + \vec{\mu} \right) \\
&= \frac{1}{2} \sum_{i=x,y,z} I_i \left(\omega_i^{(\text{rel})} + \mu_i \right)^2 = \frac{1}{2} A \left(\omega_x^{(\text{rel})} + \mu_x \right)^2 \\
&\quad + \frac{1}{2} B \left(\omega_y^{(\text{rel})} + \mu_y \right)^2 + \frac{1}{2} C \left(\omega_z^{(\text{rel})} + \mu_z \right)^2,
\end{aligned}
\tag{30}
$$

where $I_i \equiv (A, B, C)$ are the principal values of the inertia matrix of the body, $\vec{\omega}^{(\text{inert})}$ is the inertial angular velocity (i.e., the one with respect to an inertial frame), $\vec{\omega}^{(\text{rel})}$ is the relative angular velocity (i.e., the one with respect to a precessing coordinate system), while $\vec{\mu}$ is the rotation rate of the precessing frame with respect to the inertial frame. In (30), both $\vec{\omega}$'s and $\vec{\mu}$ are resolved into their components along the principal axes $\hat{\mathbf{b}}_1 \equiv \hat{\mathbf{x}}, \hat{\mathbf{b}}_2 \equiv \hat{\mathbf{y}}, \hat{\mathbf{b}}_3 \equiv \hat{\mathbf{z}}$ of the rotating body. Expression (30) is fundamental and stays, no matter whether $\vec{\mu}$ depends on the rotator's orientation, or whether it carries a direct time dependence.

The role of canonical coordinates will be played the Euler angles[18]

$$
q_n = (h_f, I_f, \phi_f)
\tag{31}
$$

that map the *precessing* coordinate basis into the principal body basis. To compute their conjugate momenta, let us assume that noninertiality of the precessing coordinate system is the only angular-velocity-dependent perturbation. Then the momenta are simply the derivatives of the kinetic energy. With aid of the formulae for the body-frame components of the relative angular velocity,[19]

---

[18] We would once again remind that the Euler angles, though normally termed (ϕ, θ, ψ), in the astronomical literature are often denoted as (ψ, θ, ϕ). In the Kinoshita–Souchay theory notations (h, I, ϕ) are employed. The angles defining orientation of the Earth's figure and of the Earth's angular-velocity vector are accompanied with the subscripts f and r, correspondingly: (h_f, I_f, ϕ_f) and (h_r, I_r, ϕ_r). The directional angles of the inertial angular velocity will be denoted with $(h_r^{(\text{inert})}, I_r^{(\text{inert})}, \phi_r^{(\text{inert})})$. Those of the relative velocity will be called $(h_r^{(\text{rel})}, I_r^{(\text{rel})}, \phi_r^{(\text{rel})})$.

[19] It should be emphasised, that the components of the angular velocity $\vec{\omega}^{(\text{rel})}$ are related to the body axes, but the angular velocity itself is the relative one (i.e., that with respect to the precessing coordinate system). Our formulae (30–34) are analogous to Eqs. 2.4 in Kinoshita (1977). At the initial step of his development, Kinoshita used his Eqs. 2.4 to express the inertial angular velocity (what we call $\vec{\omega}^{(\text{inert})}$) via the Euler angles introduced in an inertial frame. Then, on having introduced a precessing orbital frame, Kinoshita employed these equations for expressing the angular velocity through the Euler angles introduced in a precessing frame. Kinoshita did not explore whether this operation would furnish the relative angular velocity (what we call $\vec{\omega}^{(\text{rel})}$) or still the inertial one.

$$\omega_x^{(\text{rel})} = \dot{h}_f \sin I_f \sin \phi_f + \dot{I}_f \cos \phi_f, \tag{32}$$

$$\omega_y^{(\text{rel})} = \dot{h}_f \sin I_f \cos \phi_f - \dot{I}_f \sin \phi_f, \tag{33}$$

$$\omega_z^{(\text{rel})} = \dot{h}_f \cos I_f + \dot{\phi}_f, \tag{34}$$

we obtain:

$$
\begin{aligned}
p_{h_f} &= \frac{\partial T_{\text{kin}}}{\partial \dot{h}_f} = A \left(\omega_x^{(\text{rel})} + \mu_x \right) \sin I_f \sin \phi_f + B \left(\omega_y^{(\text{rel})} + \mu_y \right) \sin I_f \cos \phi_f \\
&\quad + C \left(\omega_z^{(\text{rel})} + \mu_z \right) \cos I_f,
\end{aligned}
\tag{35}
$$

$$p_{I_f} = \frac{\partial T_{\text{kin}}}{\partial \dot{I}_f} = A \left(\omega_x^{(\text{rel})} + \mu_x \right) \cos \phi_f - B \left(\omega_y^{(\text{rel})} + \mu_y \right) \sin \phi_f. \tag{36}$$

$$p_{\phi_f} = \frac{\partial T_{\text{kin}}}{\partial \dot{\phi}_f} = C \left(\omega_z^{(\text{rel})} + \mu_z \right), \tag{37}$$

These formulae enable one to express the angular-velocity components ω_i and the derivatives $\dot{q}_n = (\dot{h}_f, \dot{I}_f, \dot{\phi}_f)$ via the momenta $p_n = (p_{h_f}, p_{I_f}, p_{\phi_f})$. Insertion of (35–37) into

$$\mathcal{H} = \sum_{n=1}^{3} \dot{q}_n p_n - \mathcal{L} = \dot{h}_f p_{h_f} + \dot{I}_f p_{I_f} + \dot{\phi}_f p_{\phi_f} - T + V(h_f, I_f, \phi_f; t) \tag{38}$$

results, after some tedious algebra, in

$$\mathcal{H} = T + \Delta \mathcal{H}, \tag{39}$$

the perturbation $\Delta \mathcal{H}$ consisting of a potential term V (presumed to depend only upon the time and the angular coordinates, not upon the momenta) and a precession-generated inertial term E:

$$\Delta \mathcal{H} = V(h_f, I_f, \phi_f; t) + E,$$

where

$$
\begin{aligned}
E &= -\mu_x \left[\frac{\sin \phi_f}{\sin I_f} \left(p_{h_f} - p_{\phi_f} \cos I_f \right) + p_{I_f} \cos \phi_f \right] \\
&\quad -\mu_y \left[\frac{\cos \phi_f}{\sin I_f} \left(p_{h_f} - p_{\phi_f} \cos I_f \right) - p_{I_f} \sin \phi_f \right] - \mu_z p_{\phi_f},
\end{aligned}
\tag{40}
$$

expression equivalent to formulae (24–25) in Giacaglia and Jefferys (1971).[20] Now let us employ the machinery set out in Sect. 2.2. The fact that the Andoyer elements are introduced

---

[20] It can be shown (Gurfil et al. 2007) that the body-frame-related components g_i of the angular momentum are connected with the Euler angles and their conjugate momenta through

$$g_1 = p_{h_f} \frac{\sin \phi}{\sin I} + p_{I_f} \cos \phi - p_{\phi_f} \sin \phi \cot I, \quad g_2 = p_{h_f} \frac{\cos \phi}{\sin I} - p_{I_f} \sin \phi - p_{\phi_f} \cos \phi \cot I,$$
$$g_3 = p_{\phi_f},$$

whence it can be seen that (40) is merely another form of the relation $\Delta \mathcal{H} = -\vec{\mu} \cdot \vec{G}$. The latter can also be expressed via the Andoyer variables:

$$\Delta \mathcal{H} = -\vec{\mu} \cdot \vec{G} = -\mu_1 \sqrt{G^2 - L^2} \sin l - \mu_2 \sqrt{G^2 - L^2} \cos l - \mu_3 L.$$

in a noninertial frame is accounted for by the emergence of the μ-terms in the expression (38) for the disturbance $\Delta\mathcal{H}$. Insertion of (40) into (20) entails:

$$\dot{q}_n = g_n + \frac{\partial \Delta\mathcal{H}}{\partial p_n}, \tag{41}$$

where $q_n \equiv h_f, I_f, \phi_f$, and the convective terms are given by

$$\frac{\partial \Delta\mathcal{H}}{\partial p_{h_f}} = -\frac{\mu_x \sin\phi_f + \mu_y \cos\phi_f}{\sin I_f}, \tag{42}$$

$$\frac{\partial \Delta\mathcal{H}}{\partial p_{I_f}} = -\mu_x \cos\phi_f + \mu_y \sin\phi_f, \tag{43}$$

$$\frac{\partial \Delta\mathcal{H}}{\partial p_{\phi_f}} = \left(\mu_x \sin\phi_f + \mu_y \cos\phi_f\right) \cot I_f - \mu_z, \tag{44}$$

where $\dot{q}_n$ stand for $\dot{h}_f, \dot{I}_f, \dot{\phi}_f$, and p_n signify the corresponding momenta, while μ_x, μ_y, μ_z are the components of $\vec{\mu}$ in the principal axes of the body.

A1.2 The physical interpretation of the Andoyer variables defined in a precessing frame

The physical content of the Andoyer construction built in an inertial frame is transparent: see Fig. 3 and explanation thereto. Will all the Andoyer variables and the auxiliary angles I and J retain the same physical meaning if we re-introduce the Andoyer construction in a noninertial frame? The answer is affirmative, because a transition to a noninertial frame is no different from any other perturbation: precession of the fiducial frame $(\hat{\mathbf{s}}_1, \hat{\mathbf{s}}_2, \hat{\mathbf{s}}_3)$ is equivalent to emergence of an extra perturbing torque, one generated by the inertial forces (i.e., by the fictitious forces emerging in the noninertial frame of references). In the original Andoyer construction assembled in an inertial space, the invariable plane was orthogonal to the instantaneous direction of the angular-momentum vector: if the perturbing torques were to instantaneously vanish, the angular-momentum vector (and the invariable plane orthogonal thereto) would freeze in their positions relative to the fiducial axes $(\hat{\mathbf{s}}_1, \hat{\mathbf{s}}_2, \hat{\mathbf{s}}_3)$ (which were inertial and therefore indifferent to vanishing of the perturbation). Now, that the Andoyer construction is built in a precessing frame, the fiducial plane is no longer inertial. Nevertheless if the inertial torques were to instantaneously vanish, then the invariable plane would still freeze relative to the fiducial plane (because the fiducial plane would seize its precession). Therefore, all the variables retain their initial meaning. In particular, the variables I and J defined as above will be the angles that the angular-momentum makes, correspondingly, with the precessing $\hat{\mathbf{s}}_3$ space axis and with the $\hat{\mathbf{b}}_3$ principal axis of the body.[21] Among other things, this explains why Laskar and Robutel (1993) and Touma and Wisdom (1993, 1994), who explored the history of the Martian obliquity, arrived to very close results. Both groups rightly used the angle I as an approximation for the obliquity. While Touma and Wisdom (1993, 1994) employed (a somewhat simplified version of) the Kinoshita formalism in an inertial frame, Laskar and Robutel (1993) used this machinery in a precessing frame of the orbit. Now we understand why they obtained so close results, with minor differences

---

[21] On the interrelation between the Andoyer variables, referred to an inertial frame, and those referred to a moving frame see Eq. 3.3 in Kinoshita (1977).

stemming, most likely, from averaging-caused error accumulation in the latter paper. (The computation by Touma and Wisdom was based on unaveraged equations of motion, while Laskar and Robutel employed orbit-averaged equations.)

A1.3 Calculation of the angular velocities via the Andoyer variables introduced in a precessing frame of reference

Let us now have a look at the well-known expressions

$$\omega_x^{(\mathrm{rel})} = \dot{h}_f \sin I_f \sin \phi_f + \dot{I}_f \cos \phi_f, \tag{45}$$

$$\omega_y^{(\mathrm{rel})} = \dot{h}_f \cos \phi_f \sin I_f - \dot{I}_f \sin \phi_f, \tag{46}$$

$$\omega_z^{(\mathrm{rel})} = \dot{\phi}_f + \dot{h}_f \cos I_f, \tag{47}$$

for the principal-axes components of the precessing-frame-related angular velocity $\vec{\omega}^{(\mathrm{rel})}$. These formulae render this angular velocity as a function of the rates of Euler angle's evolution, so one can symbolically denote the functional dependence (45–47) as $\vec{\omega} = \vec{\omega}(\dot{q})$. This dependence is linear, so insertion of (41) therein will yield:

$$\vec{\omega}\,(\dot{q}(A)) = \vec{\omega}(\mathrm{g}(A)) + \vec{\omega}\,(\partial \Delta \mathcal{H}/\partial p) \tag{48}$$

with A denoting the set of Andoyer variables, and p signifying the canonical momenta corresponding to the Euler angles. Direct substitution of (42–44) into (45–47) will then show that the second term on the right-hand side in (48) is exactly $-\vec{\mu}$:

$$\vec{\omega}(\dot{q}(A)) = \vec{\omega}(\mathrm{g}(A)) - \vec{\mu}. \tag{49}$$

Since the $\vec{\omega}(\dot{q})$ is, ab initio, the relative angular velocity $\vec{\omega}^{(\mathrm{rel})}$ (i.e., that of the body frame relative to the precessing frame), and since $\vec{\mu}$ is the precession rate of that frame with respect to the inertial one, then $\vec{\omega}(\mathrm{g}(A))$ will always return the inertial angular velocity of the body, $\vec{\omega}^{(\mathrm{inert})}$ (i.e., the angular velocity relative to the inertial frame). It will do so even despite the fact that now the Andoyer parameterisation is introduced in a precessing coordinate frame! In brief, the above line of reasoning may be summarised as:

$$\left.\begin{array}{rl} \vec{\omega}\,(\dot{q}(A)) &= \vec{\omega}(\mathrm{g}(A)) + \vec{\omega}\,(\partial \Delta \mathcal{H}/\partial p)\,, \\ \vec{\omega}(\dot{q}(A)) &= \vec{\omega}^{(\mathrm{rel})}, \\ \vec{\omega}(\partial \Delta \mathcal{H}/\partial p) &= \vec{\mu}, \\ \vec{\omega}^{(\mathrm{rel})} &= \vec{\omega}^{(\mathrm{inert})} - \vec{\mu}. \end{array}\right\} \implies \vec{\omega}(\mathrm{g}(A)) = \vec{\omega}^{(\mathrm{inert})} \tag{50}$$

where the entities are defined as follows:

$\vec{\omega}^{(\mathrm{rel})}$ ≡ the relative angular velocity, i.e., the body's angular velocity relative to a precessing orbital frame;

$\vec{\mu}$ ≡ the precession rate of the orbital frame with respect to an inertial one;

$\vec{\omega}^{(\mathrm{inert})}$ ≡ the inertial angular velocity, i.e., the body's angular velocity with respect to the inertial frame.

This development parallels a situation in orbital dynamics. There the role of canonical elements is played by the Delaunay set $C = (Q; P) = (L, G, H; -M_o, -\omega, -\Omega)$ with

$$L \equiv \mu^{1/2} a^{1/2}, \quad G \equiv \mu^{1/2} a^{1/2} \left(1 - e^2\right)^{1/2}, \quad H \equiv \mu^{1/2} a^{1/2} \left(1 - e^2\right)^{1/2} \cos i,$$

the parameters $a, e, i, \omega, \Omega, M_o$ being the Kepler orbital elements. In the unperturbed setting (the two-body problem in inertial axes), the Cartesian coordinates $\vec{r} \equiv (x_1, x_2, x_3)$ and velocities $(\dot{x}_1, \dot{x}_2, \dot{x}_3)$ are expressed via the time and the Delaunay constants by means of the following functional dependencies:

$$\vec{r} = \vec{f}(C, t) \quad \text{and} \quad \vec{v} = \vec{g}(C, t), \quad \text{where } \vec{g} \equiv \partial \vec{f}/\partial t. \tag{51}$$

If we want to describe a satellite orbiting a precessing oblate planet, we may fix our reference frame on the precessing equator of date. Then the two-body problem will get amended with two disturbances. One, $\Delta \mathcal{H}_{\text{oblate}}$, caused by the presence of the equatorial bulge of the planet, will depend only upon the satellite's position. Another one, $\Delta \mathcal{H}_{\text{precess}}$, will stem from the noninertial nature of our frame and, thus, will give birth to velocity-dependent inertial forces. Under these perturbations, the Delaunay constants (now introduced in the precessing frame) will become canonical variables evolving in time. As explained in Sect. 2.3, the velocity-dependence of one of the perturbations involved will make the Delaunay variables nonosculating (provided that we keep them canonical). On the one hand, the expression $\vec{r} = \vec{f}(C(t), t)$ will return the correct Cartesian coordinates of the satellite in the precessing equatorial frame, i.e., in the frame wherein the Delaunay variables were introduced. On the other hand, the expression $\vec{g}(C, t)$ will no longer return the correct velocities in that frame. Indeed, according to (9–10), the Cartesian components of the velocity in the precessing equatorial frame will be given by $\vec{g}(C, t) + \partial \Delta \mathcal{H}_{\text{precess}}/\partial p$. However, since the second term of this sum is equal to $-\vec{\mu} \times \vec{r}$, then $\vec{g}(C, t)$ turns out to always render the velocity with respect to the inertial frame of reference (Efroimsky and Goldreich 2004; Efroimsky 2005).

Appendix 2: The instantaneous angular velocity in the Kinoshita–Souchay theory

The main burden of Sect. 2.3 in the text above was to highlight the need to add the convective term Φ to the unperturbed velocity g, in order to obtain the full velocity $\dot{q}$ under disturbance. Here q stands for a vector consisting of the three Eulerian angles $q_n = h_f, I_f, \phi_f$ defining the orientation of the principal axes of the Earth relative to the precessing frame. The corresponding convective terms, entering the expressions for $\dot{q}_n = \dot{h}_f, \dot{I}_f, \dot{\phi}_f$, are given by formula (20). Our eventual goal will be to calculate the corresponding corrections to the Euler angles determining the instantaneous axis of rotation in a precessing frame of reference.

A2.1 The unperturbed velocities

In this subsection we shall write the unperturbed Euler angles' partial time derivatives $g_n \equiv \partial q_n/\partial t$ as functions of these angles and of the Andoyer variables.

In the Kinoshita–Souchay theory of Earth rotation, the Euler angles defining the figure of the Earth are denoted with

$$q_n = (h_f, I_f, \phi_f), \tag{52}$$

the subscript standing for "figure."

Now, let us denote the principal body axes with 1, 2, 3 and the appropriate moments of inertia with A, B, C (so that $A \leq B \leq C$). The angular momentum $\vec{L}$ is connected with

the Earth-figure Euler angles via the body-frame components (45–47) of the inertial-frame-related[22] angular velocity $\vec{\omega}^{(\text{inert})}$:

$$L_x = A\omega_x^{(\text{inert})} = A\left(\dot{h}_f \sin I_f \sin \phi_f + \dot{I}_f \cos \phi_f\right), \tag{53}$$

$$L_y = B\omega_y^{(\text{inert})} = B\left(\dot{h}_f \sin I_f \cos \phi_f - \dot{I}_f \sin \phi_f\right), \tag{54}$$

$$L_z = C\omega_z^{(\text{inert})} = C\left(\dot{h}_f \cos I_f + \dot{\phi}_f\right). \tag{55}$$

On the other hand, the body-frame components of the angular momentum will be related to the Andoyer elements through[23]

$$L_x = \sqrt{G^2 - L^2}\, \sin l, \tag{56}$$

$$L_y = \sqrt{G^2 - L^2}\, \cos l, \tag{57}$$

$$L_z \equiv L. \tag{58}$$

Substituting (56–58) into (53–55) and solving for the rates of change of the Euler angles will entail:

$$\frac{\partial h_f}{\partial t} = \frac{1}{\sin I_f}\left[\frac{L_x}{A}\sin\phi_f + \frac{L_y}{B}\cos\phi_f\right]$$
$$= \frac{1}{\sin I_f}\sqrt{G^2 - L^2}\left[\frac{\sin l \sin\phi_f}{A} + \frac{\cos l \cos\phi_f}{B}\right], \tag{59}$$

$$\frac{\partial I_f}{\partial t} = \frac{L_x}{A}\cos\phi_f - \frac{L_y}{B}\sin\phi_f$$
$$= \sqrt{G^2 - L^2}\left[\frac{\sin l \cos\phi_f}{A} - \frac{\cos l \sin\phi_f}{B}\right], \tag{60}$$

$$\frac{\partial\phi_f}{\partial t} = \frac{L_z}{C} - \cot I_f\left[\frac{L_x}{A}\sin\phi_f + \frac{L_y}{B}\cos\phi_f\right]$$
$$= \frac{L}{C} - \sqrt{G^2 - L^2}\cot I_f\left[\frac{\sin l \sin\phi_f}{A} + \frac{\cos l \cos\phi_f}{B}\right], \tag{61}$$

where we deliberately replaced $\dot{h}_f, \dot{I}_f, \dot{\phi}_f$ with $\partial h_f/\partial t, \partial I_f/\partial t, \partial\phi_f/\partial t$, because so far we have been considering the situation of no disturbances turned on (i.e., the case when the full derivatives coincide with the partial ones, and lack convective terms). Our next step will be to turn on the disturbance $\Delta\mathcal{H}$, which will include a transition from an inertial frame to the precessing frame of the Earth's orbit. In accordance with formulae (2.3) and (16), this transition will generate additions to the derivatives (59–61), the additions that make the difference between a total and a partial derivative.

---

[22] Be mindful that in formulae (45–47) the notations h_f, I_f, ϕ_f stood for the Euler angles defining the body's orientation relative to a precessing frame. For this reason, (45–47) furnished the relative angular velocity $\vec{\omega}^{(\text{rel})}$. In this subsection we are beginning with the unperturbed situation, when the orbit frame is yet assumed to be inertial. Hence, at this moment, h_f, I_f, ϕ_f yet denote the angles relative to the inertial frame, and hence the same formulae render the inertial angular velocity $\vec{\omega}^{(\text{inert})}$.

[23] At this point, we are discussing the unperturbed case, with no frame precession. However, as explained in Sect. A1.2, the interconnection (56–58) between the Andoyer elements and the components of $\vec{\mathbf{L}}$ will stay valid also when the precession is "turned on" (and both the elements and the components of $\vec{\mathbf{L}}$ are introduced in a precessing frame of reference).

A2.2 Turning on the perturbation—switching to a precessing frame

Our goal here is to derive the convective terms $\Phi_n = (\Phi_{h_f}, \Phi_{I_f}, \Phi_{\phi_f})$ that are to be added to the partial derivatives (59–61), to get the full time derivatives $\dot{q}_n = (\dot{h}_f, \dot{I}_f, \dot{\phi}_f)$.

A2.2.1 Generalities

As explained by Kinoshita (1977), the undisturbed dependence of the Euler angles of the Earth's figure upon the Andoyer elements can be approximated with

$$h_f = h + \frac{J}{\sin I} \sin g + O(J^2), \tag{62}$$

$$I_f = I + J \cos g + O(J^2), \tag{63}$$

$$\phi_f = l + g - J \cot I \sin g + O(J^2), \tag{64}$$

J and I being the angles that the invariable plane (the one orthogonal to the angular momentum $\bar{G}$) makes with the body equator and with the ecliptic plane of date, correspondingly. (For the Earth, J is of order 10^{-6}.) As evident from Fig. 3, these angles are interconnected with the Andoyer variables L and G through formulae

$$L = G \cos J \tag{65}$$

and

$$H = G \cos I. \tag{66}$$

Under perturbations, formulae (62–64) will stay valid. However, the expressions for the angles' evolution rate, (59–61), will acquire convective additions (20) caused by the loss of osculation. These additions, entering the expressions for $\dot{q}_n = (\dot{h}_f, \dot{I}_f, \dot{\phi}_f)$, will read, accordingly, as

$$\Phi_{h_f} = \frac{\partial \Delta \mathcal{H}}{\partial p_{h_f}}, \quad \Phi_{I_f} = \frac{\partial \Delta \mathcal{H}}{\partial p_{I_f}}, \quad \Phi_{\phi_f} = \frac{\partial \Delta \mathcal{H}}{\partial p_{\phi_f}}. \tag{67}$$

So our next step will be to calculate these three terms.

Among the perturbations entering the Kinoshita theory, there is a so-called "E term." It emerges due to a transition from an inertial frame to a noninertial one, i.e., from a coordinate system associated with the ecliptic of epoch to the one associated with the ecliptic of date. Simply speaking, in the Kinoshita theory the Earth rotation is considered in a noninertial frame of the terrestrial orbit precessing about the Sun. In Kinoshita (1977), the xy plane of this noninertial frame is referred to as the *moving plane*. In his theory, this "E term" is the only one dependent not only upon the instantaneous orientation but also upon the angular velocity of the Earth (or, in the Hamiltonian formulation, upon the momenta conjugate to the Euler angles of the Earth's figure). Hence, in this situation $\partial \Delta \mathcal{H}/\partial p_j = \partial E/\partial p_j$. The expressions for $\Delta \mathcal{H}$ and the "E term" are rendered by formulae (39–40) where $p_{h_f}, p_{I_f}, p_{\phi_f}$ denote the canonical momenta, while μ_x, μ_y, μ_z signify the body-frame components of the angular rate at which the orbit plane is precessing relative to an inertial coordinate system.[24]

---

[24] We would point out that in Kinoshita's theory the origin of both Π_1 and h is the mean equinox, whereas in our formalism the origin simply coincides with the x axis. For our Π_1 and h to coincide with those of Kinoshita, not only must we choose our inertial coordinate system with its xy plane being within the ecliptic

In order to continue, we need the expressions for the body-frame components μ_x, μ_y, μ_z. These can be obtained from the precessing-frame components μ_1, μ_2, μ_3 by means of the appropriate rotation matrix:

$$
\begin{bmatrix} \mu_x \\ \mu_y \\ \mu_z \end{bmatrix} = \begin{bmatrix} \cos\phi_f \cos h_f & \cos\phi_f \sin h_f & \sin\phi_f \sin I_f \\ -\sin\phi_f \cos I_f \sin h_f & +\sin\phi_f \cos I_f \cos h_f & \\ -\sin\phi_f \cos h_f & -\sin\phi_f \sin h_f & \cos\phi_f \sin I_f \\ -\cos\phi_f \cos I_f \sin h_f & +\cos\phi_f \cos I_f \cos h_f & \\ \sin I_f \sin h_f & -\sin I_f \cos h_f & \cos I_f \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix}
$$

(68)

Since no other contributions in $\Delta\mathcal{H}$ other than E depend upon the momenta, then

$$
\begin{aligned}
\Phi_{h_f} &= \frac{\partial \Delta\mathcal{H}}{\partial p_{h_f}} = \frac{\partial E}{\partial p_{h_f}} = -\frac{\sin\phi_f}{\sin I_f}\mu_x - \frac{\cos\phi_f}{\sin I_f}\mu_y \\
&= \mu_1 \cot I_f \sin h_f - \mu_2 \cot I_f \cos h_f - \mu_3,
\end{aligned}
$$

(69)

$$
\begin{aligned}
\Phi_{I_f} &= \frac{\partial \Delta\mathcal{H}}{\partial p_{I_f}} = \frac{\partial E}{\partial p_{I_f}} = -\mu_x \cos\phi_f + \mu_y \sin\phi_f \\
&= -\mu_1 \cos h_f - \mu_2 \sin h_f,
\end{aligned}
$$

(70)

$$
\begin{aligned}
\Phi_{\phi_f} &= \frac{\partial \Delta\mathcal{H}}{\partial p_{\phi_f}} = \frac{\partial E}{\partial p_{\phi_f}} = \frac{\sin\phi_f \cos I_f}{\sin I_f}\mu_x + \frac{\cos\phi_f \cos I_f}{\sin I_f}\mu_y - \mu_z \\
&= -\frac{\sin h_f}{\sin I_f}\mu_1 + \frac{\cos h_f}{\sin I_f}\mu_2.
\end{aligned}
$$

(71)

Naturally, none of the Φ terms bears dependence upon ϕ_f.

To write down the Φ terms as functions of the longitude Π_1 and inclination π_1 of the ecliptic of date on that of epoch, we shall insert into (69–71) the appropriate expressions for μ_1, μ_2, μ_3. However, at this point care is needed, because of the freedom of choice of a coordinate system co-precessing with the orbital plane.[25]

A2.2.2 The precession rate $\vec{\mu}$ as seen in a certain coordinate system associated with the precessing equator of date

Let the inertial axes (X, Y, Z) be fixed in space so that X and Y belong to the ecliptic of epoch. A rotation within the ecliptic-of-epoch plane by longitude Π_1, from the axis X, will define the line of nodes. A rotation about this line by an inclination angle π_1 will give us the ecliptic of date. The line of nodes, 1, along with axis 2 naturally chosen within the ecliptic-of-date plane, and with axis 3 orthogonal to this plane, will constitute the precessing coordinate system, with the appropriate basis denoted by $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$. For example, the unit vector $\hat{e}_3$ reads in the inertial axes (X, Y, Z) as

$$
\hat{e}_3 = (\sin\pi_1 \sin\Pi_1, -\sin\pi_1 \cos\Pi_1, \cos\pi_1)^T.
$$

(72)

Footnote 24 continued

of epoch, but we should also choose the x axis to coincide with the mean equinox of epoch. Similarly, not only should our precessing frame to be associated with the ecliptic of date, but the precessing x axis whence we reckon the angles should be placed exactly at the angular distance of $-\Pi_1$ from the node—see Fig. 2 in Kinoshita (1977). (Mind that in the presence of precession Kinoshita employs notation h' instead of h.)

[25] We are grateful to Hiroshi Kinoshita who explained to us the choice accepted in his works.

The Earth's angular velocity relative to the inertial and precessing axes obey

$$\vec{\omega}^{(\text{inert})} = \vec{\omega}^{(\text{rel})} + \vec{\mu}, \tag{73}$$

$\vec{\mu}$ being the precession rate of the precessing axes $\hat{e}_j$ relative to the inertial axes (X, Y, Z). In the inertial axes, this rate is given by

$$\vec{\mu}' = \left(\dot{\pi}_1 \cos \Pi_1, \dot{\pi}_1 \sin \Pi_1, \dot{\Pi}_1\right)^T, \tag{74}$$

because this expression satisfies the equality $\vec{\mu}' \times \hat{e}_3 = \dot{\hat{e}}_3$, as can be easily seen from (72) and (74).

In a frame precessing with the ecliptic, the precession rate will be represented by the vector

$$\vec{\mu} = \hat{\mathbf{R}}_{e \to d} \vec{\mu}', \tag{75}$$

where

$$\hat{\mathbf{R}}_{e \to d} = \hat{\mathbf{R}}_1(\pi_1)\hat{\mathbf{R}}_Z(\Pi_1) = \begin{bmatrix} \cos \Pi_1 & \sin \Pi_1 & 0 \\ -\cos \pi_1 \sin \Pi_1 & \cos \pi_1 \cos \Pi_1 & \sin \pi_1 \\ \sin \pi_1 \sin \Pi_1 & -\sin \pi_1 \cos \Pi_1 & \cos \pi_1 \end{bmatrix} \tag{76}$$

is the matrix of rotation from the ecliptic of epoch to that of date. From (75–76) we get the components of the precession rate,[26] as seen in the co-precessing coordinate frame (1, 2, 3):

$$\vec{\mu} = (\mu_1, \mu_2, \mu_3)^T = \left(\dot{\pi}_1, \dot{\Pi}_1 \sin \pi_1, \dot{\Pi}_1 \cos \pi_1\right)^T. \tag{77}$$

Substitution of these components into (69–71) entails:

$$\Phi_{h_f} = \dot{\pi}_1 \cot I_f \sin h_f - \dot{\Pi}_1 \sin \pi_1 \cot I_f \cos h_f - \dot{\Pi}_1 \cos \pi_1, \tag{78}$$

$$\Phi_{I_f} = -\dot{\pi}_1 \cos h_f - \dot{\Pi}_1 \sin \pi_1 \sin h_f, \tag{79}$$

$$\Phi_{\phi_f} = -\frac{\sin h_f}{\sin I_f}\dot{\pi}_1 + \frac{\cos h_f}{\sin I_f}\dot{\Pi}_1 \sin \pi_1. \tag{80}$$

A2.2.3 The precession rate $\vec{\mu}$ as seen in a different coordinate system associated with the precessing equator of date (the system used by Kinoshita 1977)

In the preceding subsection the transition from the ecliptic of epoch to the one of date was implemented by two Euler rotations: $\hat{\mathbf{R}}_{e \to d} = \hat{\mathbf{R}}_N(\pi_1)\hat{\mathbf{R}}_Z(\Pi_1) = \hat{\mathbf{R}}_1(\pi_1)\hat{\mathbf{R}}_Z(\Pi_1)$. The axis 1 of the precessing frame was assumed to coincide with the line of nodes, N. Evidently, this choice was just one out of an infinite multitude. An alternative option was employed by Kinoshita (1977), who used a sequence of three Eulerian rotations: $\hat{\mathbf{R}}_{e \to d}^K = \hat{\mathbf{R}}_3(-\Pi_1)\hat{\mathbf{R}}_N(\pi_1)\hat{\mathbf{R}}_Z(\Pi_1)$. Specifically, having performed the two rotations described above, Kinoshita then rotated the axis 1 within the ecliptic of date by angle $-\Pi_1$ away from the line of nodes N. Due to reasoning analogous to what was presented in the subsection above, the

---

[26] Equivalently, one can find the components of $\vec{\mu}$ as the elements of the skew-symmetric matrix $\dot{\hat{\mathbf{R}}}_{e \to d}\hat{\mathbf{R}}_{e \to d}^{-1}$.

sequence of three rotations gives, instead of (77), the following expression:[27]

$$\vec{\mu} = (\mu_1, \mu_2, \mu_3)^T$$
$$= (\dot{\pi}_1 \cos \Pi_1 - \dot{\Pi} \sin \Pi_1 \sin \pi_1, \ \dot{\pi}_1 \sin \Pi_1 + \dot{\Pi}_1 \cos \Pi_1 \sin \pi_1,$$
$$\dot{\Pi}_1 \cos \pi_1 - \dot{\Pi}_1)^T. \tag{81}$$

Insertion thereof into (69–71) will yield:

$$\Phi_{h_f} = \dot{\pi}_1 \cot I_f \sin(h_f - \Pi_1) - \dot{\Pi}_1 \sin \pi_1 \cot I_f \cos(h_f - \Pi_1)$$
$$- \dot{\Pi}_1 \cos \pi_1 + \Pi_1, \tag{82}$$

$$\Phi_{I_f} = -\dot{\pi}_1 \cos(h_f - \Pi_1) - \dot{\Pi}_1 \sin \pi_1 \sin(h_f - \Pi_1), \tag{83}$$

$$\Phi_{\phi_f} = -\frac{\sin(h_f - \Pi_1)}{\sin I_f} \dot{\pi}_1 + \frac{\cos(h_f - \Pi_1)}{\sin I_f} \dot{\Pi}_1 \sin \pi_1. \tag{84}$$

A2.3 The perturbed velocities

According to (18), to get the full evolution of the figure-axis Euler angles under perturbation, one should sum the unperturbed velocities, given by the partial derivatives (69–71), with the appropriate convective terms (69–71):

$$\dot{h}_f = \left(\frac{\partial h_f}{\partial t}\right)_C + \Phi_{h_f} = \frac{1}{\sin I_f} \sqrt{G^2 - L^2} \left[\frac{\sin l \sin \phi_f}{A} + \frac{\cos l \cos \phi_f}{B}\right]$$
$$+ \dot{\pi}_1 \cot I_f \sin h_f - \dot{\Pi}_1 \sin \pi_1 \cot I_f \cos h_f - \dot{\Pi}_1 \cos \pi_1, \tag{85}$$

$$\dot{I}_f = \left(\frac{\partial I_f}{\partial t}\right)_C + \Phi_{I_f} = \sqrt{G^2 - L^2} \left[\frac{\sin l \cos \phi_f}{A} - \frac{\cos l \sin \phi_f}{B}\right]$$
$$- \dot{\pi}_1 \cos h_f - \dot{\Pi}_1 \sin \pi_1 \sin h_f, \tag{86}$$

$$\dot{\phi}_f = \left(\frac{\partial \phi_f}{\partial t}\right)_C + \Phi_{\phi_f} = \frac{L}{C} - \sqrt{G^2 - L^2} \cot I_f \left[\frac{\sin l \sin \phi_f}{A} + \frac{\cos l \cos \phi_f}{B}\right]$$
$$- \frac{\sin h_f}{\sin I_f} \dot{\pi}_1 + \frac{\cos h_f}{\sin I_f} \dot{\Pi}_1 \sin \pi_1. \tag{87}$$

These expressions will help us to determine the instantaneous orientation of the spin axis.

A2.4 The precessing-frame-related angular velocity expressed through the Andoyer elements introduced in the precessing frame

Let angles $h_r^{(rel)}$ and $I_r^{(rel)}$ be the precessing-frame-related node and inclination of vector of the angular velocity relative to the precessing frame, and let $\omega^{(rel)} \equiv |\vec{\omega}^{(rel)}|$ be the relative spin rate. Our next step will be to derive expressions for $h_r^{(rel)}$ and $I_r^{(rel)}$ as functions of $h_f, I_f, \phi_f, \dot{h}_f, \dot{I}_f, \dot{\phi}_f$, and then to substitute the expressions (85–87) for $\dot{h}_f, \dot{I}_f, \dot{\phi}_f$ therein,

---

[27] In the precessing frame, the angular momentum reads: $(G \sin I \sin h, -G \sin I \cos h, H)^T$, quantities I, h, and H being as in Fig. 3. This, together with (81) and the formula $\Delta \mathcal{H} = E = -\vec{\mu} \cdot \vec{G}$, will entail:

$$\Delta \mathcal{H} = E = \dot{\Pi}_1 H (1 - \cos \pi_1) - \dot{\pi}_1 G \sin I \sin(h - \Pi_1) + \dot{\Pi}_1 G \sin I \cos(h - \Pi_1) \sin \pi_1,$$

which coincides with expression (3.4) in Kinoshita (1977).

in order to express $h_r^{(rel)}$ and $I_r^{(rel)}$ via h_f, I_f, ϕ_f only. To accomplish this step, let us begin with the formulae interconnecting the precessing-frame components of the angular velocity relative to the precessing frame with the figure-axis Euler angles and with the spin-axis Euler angles. These formulae are fundamental and perturbation-invariant:

$$\omega_1^{(rel)} = \dot{I}_f \cos h_f + \dot{\phi}_f \sin I_f \sin h_f, \tag{88}$$

$$\omega_2^{(rel)} = \dot{I}_f \sin h_f - \dot{\phi}_f \sin I_f \cos h_f, \tag{89}$$

$$\omega_3^{(rel)} = \dot{h}_f + \dot{\phi}_f \cos I_f \tag{90}$$

and

$$\omega_1^{(rel)} = \omega^{(rel)} \sin I_r^{(rel)} \sin h_r^{(rel)}, \tag{91}$$

$$\omega_2^{(rel)} = -\omega^{(rel)} \sin I_r^{(rel)} \cos h_r^{(rel)}, \tag{92}$$

$$\omega_3^{(rel)} = \omega^{(rel)} \cos I_r^{(rel)}. \tag{93}$$

Both the inertial and relative spin rates, $\omega^{(inert)}$ and $\omega^{(rel)}$, can be most conveniently calculated in the body frame. In that frame, the inertial angular velocity can be written as

$$\vec{\omega}^{(inert)} = \left(\frac{L_x}{A}, \frac{L_y}{B}, \frac{L_z}{C} \right)^T$$

$$= \left(\frac{G \sin J \sin l}{A}, \frac{G \sin J \cos l}{B}, \frac{G \cos J}{C} \right)^T, \tag{94}$$

whence its absolute value turns out to be

$$\omega^{(inert)} = \sqrt{\left(\frac{L_x}{A} \right)^2 + \left(\frac{L_y}{B} \right)^2 + \left(\frac{L_z}{C} \right)^2}$$

$$= \sqrt{\left(\frac{G \sin J \sin l}{A} \right)^2 + \left(\frac{G \sin J \cos l}{B} \right)^2 + \left(\frac{G \cos J}{C} \right)^2}$$

$$= \frac{G}{C} \left[1 + O\left(J^2 \right) \right] = \frac{L}{C} \left[1 + O\left(J^2 \right) \right]. \tag{95}$$

To derive the relative rate, square the obvious equality $\vec{\omega}^{(rel)} = \vec{\omega}^{(inert)} - \vec{\mu}$, to obtain:

$$\left(\vec{\omega}^{(rel)} \right)^2 = \left(\vec{\omega}^{(inert)} \right)^2 - 2\vec{\omega}^{(inert)} \cdot \vec{\mu} + \vec{\mu}^2. \tag{96}$$

Hence,

$$\omega^{(rel)} = \omega^{(inert)} \left[1 - \alpha + O\left((\Phi/\omega)^2 \right) \right]$$

$$= \frac{L}{C} \left[1 - \alpha + O\left((\Phi/\omega)^2 + O(J^2) \right) \right] \tag{97}$$

and

$$\frac{1}{\omega^{(rel)}} = \frac{1}{\omega^{(inert)}} \left[1 + \alpha + O\left((\Phi/\omega)^2 \right) \right]$$

$$= \frac{C}{L} \left[1 + \alpha + O\left((\Phi/\omega)^2 \right) + O(J^2) \right], \tag{98}$$

where

$$\alpha \equiv \frac{\vec{\omega}^{(\text{inert})} \cdot \vec{\mu}}{\left(\vec{\omega}^{(\text{inert})}\right)^2}. \tag{99}$$

is of order Φ/ω. Dot-multiplying (94) by (68), we arrive at:

$$\alpha = \frac{G \cos J}{C} \frac{\mu_z}{\left(\vec{\omega}^{(\text{inert})}\right)^2} + O(J\Phi/\omega)$$

$$= \frac{C}{L} \left[\mu_1 \sin I_f \sin h_f - \mu_2 \sin I_f \cos h_f + \mu_3 \cos I_f \right]$$

$$+ O(J\Phi/\omega) + O(J^2). \tag{100}$$

In the coordinate system described in A.2.2.2, substitution of (77) makes the latter read

$$\alpha = \frac{C}{L} \left(\dot{\pi}_1 \sin I_f \sin h_f - \dot{\Pi}_1 \sin \pi_1 \sin I_f \cos h_f + \dot{\Pi}_1 \cos \pi_1 \cos I_f \right)$$

$$+ O(J\Phi/\omega) + O(J^2). \tag{101}$$

In Kinoshita's coordinate system described in Sect. A.2.2.3, one should use for the components of $\vec{\mu}$ not expression (77) but (81), insertion whereof into (100) results in:

$$\alpha = \frac{C}{L} \left(\dot{\pi}_1 \sin I_f \sin(h_f - \Pi_1) - \dot{\Pi}_1 \sin \pi_1 \sin I_f \cos(h_f - \Pi_1) + \dot{\Pi}_1 \cos \pi_1 \cos I_f \right.$$

$$\left. - \dot{\Pi}_1 \cos I_f \right) + O(J\Phi/\omega) + O(J^2). \tag{102}$$

This formula will enable us to derive approximate (valid up to $O\left(J^2\right) + O\left(J\Phi/\omega\right) + O\left((\Phi/\omega)^2\right)$) expressions for $h_r^{(rel)}$ and $I_r^{(rel)}$ expressed as functions of the Andoyer variables.

Appendix 3: Expression for $I_r^{(\text{rel})}$

Together, (90) and (93) will give:

$$\omega^{(\text{rel})} \cos I_r^{(\text{rel})} = \dot{h}_f + \dot{\phi}_f \cos I_f \tag{103}$$

or, equivalently,

$$\omega^{(\text{rel})} \left(\cos I_r^{(\text{rel})} - \cos I_f \right) = \dot{h}_f + \left(\dot{\phi}_f - \omega^{(\text{rel})} \right) \cos I_f. \tag{104}$$

Since we are planning to carry out all calculations neglecting terms $O\left(J^2\right)$, and since the three inclinations I_f, $I_r^{(\text{rel})}$, I differ from one another by quantities of order $O(J)$, we can approximate the left-hand side of (104) with the first-order terms of its Taylor expansion:

$$\omega^{(\text{rel})} \left(-\sin I_f \right) \left(I_r^{(\text{rel})} - I_f \right) = \dot{h}_f + \left(\dot{\phi}_f - \omega^{(\text{rel})} \right) \cos I_f + O\left(J^2\right), \tag{105}$$

wherefrom

$$I_r^{(\text{rel})} - I_f = -\frac{\dot{h}_f}{\omega^{(\text{rel})}} \frac{1}{\sin I_f} + \left[\frac{\dot{\phi}_f}{\omega^{(\text{rel})}} - 1 \right] \left(-\cot I_f \right) + O\left(J^2\right)$$

$$= -\left(\frac{\partial h_f}{\partial t} + \Phi_{h_f} \right) \frac{1}{\omega^{(\text{rel})}} \frac{1}{\sin I_f} + \left[\left(\frac{\partial \phi_f}{\partial t} + \Phi_{\phi_f} \right) \frac{1}{\omega^{(\text{rel})}} - 1 \right] \left(-\cot I_f \right)$$

$$+ O\left(J^2\right). \tag{106}$$

To get rid of the time derivatives, employ formulae (85–87) or (59–61):

$$I_r^{(\text{rel})} - I_f = \cot I_f \left\{ 1 - \frac{1}{\omega^{(\text{rel})}} \left[\frac{L}{C} - \sqrt{G^2 - L^2} \cot I_f \left(\frac{\sin l \sin \phi_f}{A} + \frac{\cos l \cos \phi_f}{B} \right) \right] \right\}$$

$$- \frac{1}{\omega^{(\text{rel})} \sin^2 I_f} \sqrt{G^2 - L^2} \left(\frac{\sin l \sin \phi_f}{A} + \frac{\cos l \cos \phi_f}{B} \right)$$

$$- \frac{1}{\omega^{(\text{rel})} \sin I_f} \left(\Phi_{h_f} + \Phi_{\phi_f} \cos I_f \right) + O\left(J^2 \right). \tag{107}$$

For ϕ_f we can use the approximation $\phi_f = l + g - J \cot I \sin g + O(J^2)$. Besides, in the terms of order J and of order Φ/ω we can substitute $\omega^{(\text{rel})}$ with $\omega^{(\text{inert})} = L/C + O(J^2)$. Such substitutions will entail errors of orders $O(J\Phi/\omega)$ and $O((\Phi/\omega)^2)$. However, in the leading term we must use (97–101). Thus we get:

$$I_r^{(\text{rel})} - I_f = \cot I_f \left(1 - \frac{1}{\omega^{(\text{rel})}} \frac{L}{C} \right) + \frac{\sqrt{G^2 - L^2}}{\omega^{(\text{rel})}} \left(\cot^2 I_f - \frac{1}{\sin^2 I_f} \right)$$

$$\times \left[\frac{\sin l \sin \phi_f}{A} + \frac{\cos l \cos \phi_f}{B} \right] - \frac{1}{\omega^{(\text{rel})}} \frac{1}{\sin I_f} \left(\Phi_{h_f} + \Phi_{\phi_f} \cos I_f \right)$$

$$+ O(J^2)$$

$$= \cot I_f \left(1 - \frac{1 + \alpha}{\omega^{(\text{inert})}} \frac{L}{C} \right) - \frac{G \sin J}{G/C}$$

$$\times \left[\frac{\sin l}{A} \sin (l + g - J \cot I \sin g) + \frac{\cos l}{B} \cos (l + g - J \cot I \sin g) \right]$$

$$- \frac{1}{L/C} \frac{1}{\sin I_f} \left(\Phi_{h_f} + \Phi_{\phi_f} \cos I_f \right) + O\left(J^2 \right) + O\left(J\Phi/\omega \right) + O((\Phi/\omega)^2)$$

$$= -\alpha \cot I_f - CJ \left[\frac{\sin l}{A} \sin (l + g) + \frac{\cos l}{B} \cos (l + g) \right]$$

$$- \frac{C}{L} \frac{1}{\sin I_f} \left(\Phi_{h_f} + \Phi_{\phi_f} \cos I_f \right) + O\left(J^2 \right) + O\left(J\Phi/\omega \right) + O\left((\Phi/\omega)^2 \right), \tag{108}$$

whence, by using (101) and the formula $I_f = I + J \cos g + O(J^2)$, we arrive at:

$$I_r^{(\text{rel})} = I + J \left\{ \cos g - \frac{C}{A} \sin l \sin (l + g) - \frac{C}{B} \cos l \cos (l + g) \right\}$$

$$- \frac{C}{L} \frac{1}{\sin I_f} \left(\Phi_{h_f} + \Phi_{\phi_f} \cos I_f \right) + O\left(J^2 \right) + O\left(J\Phi/\omega \right) + O\left((\Phi/\omega)^2 \right)$$

Via trigonometric transformations, the second term gets simplified as:

$$\cos g - \frac{C}{A} \sin l \sin (l + g) - \frac{C}{B} \cos l \cos (l + g)$$

$$= \cos g - \frac{C}{A} \frac{\cos g - \cos(2l + g)}{2} - \frac{C}{B} \frac{\cos g + \cos(2l + g)}{2}$$

$$= \left(1 - \frac{C}{2A} - \frac{C}{2B} \right) \left[\cos g - e \cos (2l + g) \right]. \tag{109}$$

Insertion of (69–71) into the fourth term will make it look:

$$-\frac{C}{L}\frac{1}{\sin I_f}\left(\Phi_{h_f}+\Phi_{\phi_f}\cos I_f\right)=\frac{C}{L}\frac{\mu_3}{\sin I_f}. \tag{110}$$

Hence, we have for $I_r^{(\text{rel})}$:

$$I_r^{(\text{rel})}=I+J\left(1-\frac{C}{2A}-\frac{C}{2B}\right)\left[\cos g-e\cos(2l+g)\right]+\frac{C}{L}\frac{\mu_3}{\sin I_f}-\alpha\cot I_f$$
$$+O\left(J^2\right)+O(J\Phi/\omega)+O\left((\Phi/\omega)^2\right), \tag{111}$$

where the parameter e, given by (23), is the measure of triaxiality of the rotator.

In the precessing coordinate system obtained from the inertial one by two Euler rotations, as in Sect. A.2.2.2, we must now substitute (101) for α and (77) for μ_3, to get:

$$I_r^{(\text{rel})}=I+J\left(1-\frac{C}{2A}-\frac{C}{2B}\right)\left[\cos g-e\cos(2l+g)\right]-\frac{C}{L}\dot\pi_1\cos I_f\sin h$$
$$+\frac{C}{L}\dot\Pi_1\left(\sin\pi_1\cos I_f\cos h+\cos\pi_1\sin I_f\right)+O\left(J^2\right)+O(J\Phi/\omega)$$
$$+O((\Phi/\omega)^2). \tag{112}$$

In the Kinoshita precessing axes obtained from the inertial ones by three rotations, as in Sect. A.2.2.3, we should substitute (102) for α and (81) for μ_3, to get:

$$I_r^{(\text{rel})}=I+J\left(1-\frac{C}{2A}-\frac{C}{2B}\right)\left[\cos g-e\cos(2l+g)\right]$$
$$-\frac{C}{L}\dot\pi_1\cos I_f\sin(h-\Pi_1)+\frac{C}{L}\dot\Pi_1\left(\sin\pi_1\cos I_f\cos(h-\Pi_1)+\cos\pi_1\sin I_f\right.$$
$$\left.-\sin I_f\right)+O\left(J^2\right)+O(J\Phi/\omega)+O((\Phi/\omega)^2). \tag{113}$$

To arrive to the final expression for $I_r^{(\text{rel})}$, we shall make use of (62–64). These formulae will enable us to substitute, in the above expression, I_f and h_f with I and h, correspondingly. All in all, in the coordinate system as in A.2.2.2 we have:

$$I_r^{(\text{rel})}=I+J\left(1-\frac{C}{2A}-\frac{C}{2B}\right)\left[\cos g-e\cos(2l+g)\right]-\frac{C}{L}\dot\pi_1\cos I\sin h$$
$$+\frac{C}{L}\dot\Pi_1\left(\sin\pi_1\cos I\cos h+\cos\pi_1\sin I\right)+O\left(J^2\right)+O(J\Phi/\omega)$$
$$+O((\Phi/\omega)^2). \tag{114}$$

In the coordinate system as in A.2.2.3, we obtain:

$$I_r^{(\text{rel})}=I+J\left(1-\frac{C}{2A}-\frac{C}{2B}\right)\left[\cos g-e\cos(2l+g)\right]-\frac{C}{L}\dot\pi_1\cos I\sin(h-\Pi_1)$$
$$+\frac{C}{L}\dot\Pi_1\left[\sin\pi_1\cos I\cos(h-\Pi_1)+\cos\pi_1\sin I-\sin I\right]+O\left(J^2\right)$$
$$+O(J\Phi/\omega)+O((\Phi/\omega)^2). \tag{115}$$

In (114) and (115), the first two terms coincide with those given by the second of formulae (2.6) in Kinoshita (1977). They make $I_r^{(\text{inert})}$, while the third term is $I_r^{(\Phi)}$.

Appendix 4: Expression for $h_r^{(\text{rel})}$

Expressions (88) and (89) result in

$$\omega^{(\text{rel})} \sin I_r^{(\text{rel})} \sin h_r^{(\text{rel})} = \dot{I}_f \cos h_f + \dot{\phi}_f \sin I_f \sin h_f, \tag{116}$$

while (89) and (92) entail

$$-\omega^{(\text{rel})} \sin I_r^{(\text{rel})} \cos h_r^{(\text{rel})} = \dot{I}_f \sin h_f - \dot{\phi}_f \sin I_f \cos h_f. \tag{117}$$

Multiplying the former with $\cos h_f$ and the latter with $\sin h_f$, and then summing up the two results, we arrive at

$$\dot{I}_f = \omega^{(\text{rel})} \sin I_r^{(\text{rel})} \sin \left(h_r^{(\text{rel})} - h_f \right). \tag{118}$$

Since the difference $h_r^{(\text{rel})} - h_f$ is expected to be of order $O(J) + O(\Phi/\omega)$, the above formula may be rewritten as

$$h_r^{(\text{rel})} - h_f = \frac{\dot{I}_f}{\omega^{(\text{rel})} \sin I_f} + O(J^2) + O((\Phi/\omega)^2) + O(J\Phi/\omega) \tag{119}$$

or, according to (98), as

$$h_r^{(\text{rel})} - h_f = \frac{1+\alpha}{\omega^{(\text{inert})}} \frac{1}{\sin I_f} \left(\frac{\partial I_f}{\partial t} + \Phi_{I_f} \right) + O(J^2) + O((\Phi/\omega)^2) + O(J\Phi/\omega), \tag{120}$$

where α is of order $O(\Phi/\omega)$ and therefore may be dropped. Recall that, according to (95), the absolute value of the angular-velocity vector can be approximated, up to $O(J^2)$, with $\omega^{(\text{inert})} \approx L/C \approx G/C$, while $\sqrt{G^2 - L^2}$ can be expressed as $\sqrt{G^2 - L^2} = G \sin J \approx GJ \approx LJ$. Together with approximations $\phi_f = l + g - J \cot I \sin g + O(J^2)$ and $I_f = I + J \cos g + O(J^2)$, it will enable us to rewrite (60) as

$$\frac{\partial I_f}{\partial t} = LJ \left[\frac{\sin l \cos(l+g)}{A} - \frac{\cos l \sin(l+g)}{B} \right] + O(J^2), \tag{121}$$

insertion whereof in (121) will then entail

$$h_r^{(\text{rel})} - h_f = \frac{JC}{\sin I_f} \left[\frac{\sin l \cos(l+g)}{A} - \frac{\cos l \sin(l+g)}{B} \right]$$
$$+ \frac{C}{L} \frac{1}{\sin I_f} \Phi_{I_f} + O(J^2) + O((\Phi/\omega)^2) + O(J\Phi/\omega) \tag{122}$$

This, along with $h_f = h + J \sin g/\sin I + O(J^2)$ and $I_f = I + J \cos g + O(J^2)$, yields:

$$h_r^{(\text{rel})} = h + J \frac{\sin g}{\sin I} + \frac{JC}{\sin I} \left\{ \frac{\sin l}{A} \cos(l+g) - \frac{\cos l}{B} \sin(l+g) \right\} + \frac{C}{L} \frac{1}{\sin I} \Phi_{I_f} + O(\ldots)$$

$$= h + \frac{J}{\sin I} \left\{ \sin g + \frac{C}{A} \frac{\sin(2l+g) - \sin g}{2} - \frac{C}{B} \frac{\sin(2l+g) + \sin g}{2} \right\}$$
$$+ \frac{C}{L} \frac{1}{\sin I} \left(-\mu_1 \cos h_f - \mu_2 \sin h_f \right) + O(J^2) + O((\Phi/\omega)^2) + O(J\Phi/\omega)$$

$$= h + \frac{J}{\sin I} \left(1 - \frac{C}{2A} - \frac{C}{2B} \right) \left[\sin g - e \sin(2l+g) \right]$$
$$+ \frac{C}{L} \frac{1}{\sin I} \left(-\mu_1 \cos h_f - \mu_2 \sin h_f \right) + O(J^2) + O((\Phi/\omega)^2) + O(J\Phi/\omega).$$

To get the answer in the precessing coordinate axes obtained from the inertial ones by two Euler rotations, as in A.2.2.2, we substitute (77) for μ_1 and μ_2. It yields:

$$
\begin{aligned}
h_r^{(\mathrm{rel})} = \; & h + \frac{J}{\sin I}\left(1 - \frac{C}{2A} - \frac{C}{2B}\right)\left[\sin \mathrm{g} - e \sin (2l + \mathrm{g})\right] - \dot{\pi}_1 \frac{C}{L}\frac{\cos h}{\sin I} \\
& - \dot{\Pi}_1 \frac{C}{L}\frac{\sin \pi_1 \sin h}{\sin I} + O(J^2) + O((\Phi/\omega)^2) + O(J\Phi/\omega). \quad (123)
\end{aligned}
$$

To obtain the answer in the Kinoshita precessing coordinate system obtained from the inertial one by three Euler rotations, as in A.2.2.3, we should substitute (81) for μ_1 and μ_2. This will entail:

$$
\begin{aligned}
h_r^{(\mathrm{rel})} = \; & h + \frac{J}{\sin I}\left(1 - \frac{C}{2A} - \frac{C}{2B}\right)\left[\sin \mathrm{g} - e \sin (2l + \mathrm{g})\right] \\
& - \dot{\pi}_1 \frac{C}{L}\frac{\cos(h - \Pi_1)}{\sin I} - \dot{\Pi}_1 \frac{C}{L}\frac{\sin \pi_1 \sin(h - \Pi_1)}{\sin I} \\
& + O(J^2) + O((\Phi/\omega)^2) + O(J\Phi/\omega), \quad (124)
\end{aligned}
$$

the triaxiality parameter e being rendered by (23). In (123) and (124), the first two terms coincide with those given by the first expression of (2.6) in Kinoshita (1977). They constitute $h_r^{(\mathrm{inert})}$, while the third term is h_r^{Φ}.

References

Abdullah, K., Albouy, A.: On a strange resonance noticed by M. Herman. Regular and Chaotic Dynamics **6**, 421–432 (2001)

Andoyer, H.: Cours de Mécanique Céleste. Gauthier-Villars, Paris (1923)

Boccaletti, D., Pucacco, G.: Theory of Orbits. Volume 2: Perturbative and Geometrical Methods, Chapter 8. Springer Verlag, Heidelberg (2002)

Brouwer, D., Clemence, G.M.: Methods of Celestial Mechanics. Chapter XI. Academic Press, NY & London (1961)

Deprit, A.: Canonical transformations depending on a small parameter. Celest. Mech. **1**, 12–30 (1969)

Deprit, A., Elipe, A.: Complete reduction of the Euler-Poinsot problem. J Astronaut Sci **41**(4), 603–628 (1993)

Efroimsky, M.: Equations for the orbital elements. Hidden symmetry. Preprint no 1844 of the Institute of Mathematics and its Applications, University of Minnesota (2002a) http://www.ima.umn.edu/preprints/feb02/feb02.html

Efroimsky, M.: The implicit gauge symmetry emerging in the n-body problem of celestial mechanics (2002b) astro-ph/0212245

Efroimsky, M., Goldreich, P. Gauge symmetry of the N-body problem in the Hamilton–Jacobi approach. J. Math. Phys. **44**, 5958–5977 (2003) astro-ph/0305344

Efroimsky, M., Goldreich, P.: Gauge freedom in the N-body problem of celestial mechanics. Astron. Astrophys. **415**, 1187–1199 (2004) astro-ph/0307130

Efroimsky, M.: On the theory of canonical perturbations and its application to Earth rotation. Talk at the conference Journées 2004: Systèmes de référence spatio-temporels, l'Observatoire de Paris, 20–22 septembre (2004) astro-ph/0409282

Efroimsky, M.: Long-term evolution of orbits about a precessing oblate planet. The case of uniform precession. Celest. Mech. Dyn. Astron. **91**, 75–108 (2005) astro-ph/0408168

Efroimsky, M.: Gauge freedom in orbital mechanics. Ann New York Acad Sci. **1065**, 346–374 (2006) astro-ph/0603092

Escapa, A., Getino, J., Ferrándiz, J.: Canonical approach to the free nutations of a three-layer Earth model. J. Geophys. Res. **106**(B6), 11387–11397 (2001)

Escapa, A., Getino, J., Ferrándiz, J.: Indirect effect of the triaxiality in the Hamiltonian theory for the rigid Earth nutations. Astron Astrophys. **389**, 1047–1054 (2002)

Fukushima, T., Ishizaki, H.: Elements of spin motion. Celest. Mech. Dyn. Astron. **59**, 149–159 (1994)

Getino, J., Ferrándiz, J.: A Hamiltonian theory for an elastic earth. Canonical Variables and Kinetic Energy. Celest. Mech. Dyn. Astron. **49**, 303–326 (1990)

Getino, J., Ferrándiz, J.: A rigorous Hamiltonian approach to the rotation of elastic bodies. Celest. Mech. Dyn. Astron. **58**, 277–295 (1994)

Giacaglia, G.E.O., Jefferys, W.H.: Motion of a space station. I. Celest. Mech. **4**, 442–467 (1971)

Goldreich, P.: Inclination of satellite orbits about an oblate precessing planet. Astron. J. **70**, 5–9 (1965)

Goldstein, H.: Classical Mechanics. 2nd ed. Addison-Wesley, Reading MA (1981)

Gurfil, P., Elipe, A., Tangren, W., Efroimsky, M.: The Serret-Andoyer formalism in rigit-body dynamics: I. Symmetries and perturbations. Submitted to Regular and Chaotic Dynamics (2007) astro-ph/0607201

Hori, G.-I.: Theory of general perturbations with unspecified canonical variables. Publ. Astron. Soc. Jpn **18**, 287–296 (1966)

Kholshevnikov, K.V.: Lie transformations in celestial mechanics. In: Astronomy and Geodesy. Thematic Collection of Papers, **4**, Issue 4, pp. 21–45. Published by the Tomsk State University Press, Tomsk, Russia (1973) (in Russian)

Kholshevnikov, K.V.: Asymptotic Methods of Celestial Mechanics, Chapter 5. Leningrad State University Press, St.Petersburg, Russia (1985) /in Russian/

Kinoshita, H.: First-order perturbations of the two finite-body problem. Pub. Astron. Soc. Jpn. **24**, 423–457 (1972)

Kinoshita, H.: Theory of the rotation of the rigid Earth. Celest. Mech. **15**, 277–326 (1977)

Kinoshita, H., Nakajima, K., Kubo, Y., Nakagawa, I., Sasao, T., Yokoyama, K.: Note on nutation in ephemerides. Publi. Int. Latitude Observat. Mizusawa **XII**(1), 71–108 (1978)

Kinoshita, H., Souchay, J.: The theory of the nutation for the rigid-Earth model at the second order. Celest. Mech. Dyn. Astron. **48**, 187–265 (1990)

Laskar, J., Robutel, J.: The chaotic obliquity of the planets. Nature **361**, 608–612 (1993)

Lieske, J.H., Lederle, T., Fricke, W., Morando, B.: Expressions for the precession quantities based upon the IAU /1976/ system of astronomical constants. Astron. Astrophys. **58**, 1–16 (1977)

Mysen, E.: Rotational dynamics of subsolar sublimating triaxial comets. Planet. Space Sci. **52**, 897–907 (2004)

Mysen, E.: Canonical rotation variables and non-Hamiltonian forces: solar radiation pressure effects on asteroid rotation. Monthly Notices Roy. Astron. Soc. **372**, 1345–1350 (2006)

Peale, S.J.: Rotation of solid bodies in the solar system. Rev. Geophys. Space Phy. **11**, 767–793 (1973)

Peale, S.J.: Excitation and relaxation of the wobble, precession, and libration of the Moon. J Geophy. Res. **81**, 1813–1827 (1976)

Petrov, L.: The empirical Earth-rotation model from VLBI observations. Astron Astrophys. **467**, 359–369 (2007)

Plummer, H.C.: An Introductory Treatise on Dynamical Astronomy. Cambridge University Press, UK (1918)

Poincaré, H.: Sur une forme nouvelle des équations du problème des trois corps. Bull. Astron. **14**, 53–67 (1897). For modern edition see: Œuvres de Henri Poincaré, Tome **VII**, pp. 500–511. Gauthier-Villars, Paris (1950)

Radau, R.: Sur la rotation des corps solides. Annales de l'Ecole Normale Supérieure. 1$^{\text{re}}$ série. Tome **6**, 233–250 (1869) http://www.numdam.org/item?id=ASENS_1869_1_6_233_0

Richelot, F.J.: Eine neue Lœsung des Problemes der Rotation eines festen Körpers um einen Punkt. Abhandlungen der Königlichen Preuβischen Akademie der Wissenschaften zu Berlin. Math., 1–60 (1850)

Seidelmann, P.K. (ed.): Explanatory Supplement to the Astronomical Almanac. University Science Books, Mill Valley CA (1992)

Serret, J.A.: Mémoire sur l'emploi de la méthode de la variation des arbitraires dans la théorie des mouvements de rotation. Mémoires de l'Academie Des Sciences de Paris **55**, 585–616 (1866)

Schreiber, K.U., Velikoseltsev, A., Rothacher, M., Klügel, T., Stedman, G.E., Wiltshire, D.L.: Direct measurements of diurnal polar motion by ring laser gyroscopes. J. Geophys. Res. **109**, B06405 (2004)

Souchay, J., Losley, B., Kinoshita, H., Folgueira, M.: Corrections and new developments in rigid Earth nutation theory. III. Final tables "REN-2000" including crossed-nutation and spin–orbit coupling effects. Astron. Astrophys. Suppl. **135**, 111–131 (1999)

Synge, J.L., Griffith, B.A.: Principles of Mechanics. McGraw-Hill, NY (1959)

Tisserand, F.: Traité de mécanique Céleste. Gauthier-Villars, Paris (1889)

Touma, J., Wisdom, J.: The chaotic obliquity of Mars. Science **259**(5099), 1294–1297 (1993)

Touma, J., Wisdom, J.: Lie-Poisson integrators for rigid body dynamics in the solar system. Astron. J. **107**, 1189–1202 (1994)

Zanardi, M., Vilhena de Moraes, R.: Analytical and semi-analytical analysis of an artificial satellite's rotational motion. Celest. Mech. Dyn. Astron. **75**, 227–250 (1999)